Research Notes

Ethan Lu

Notes for Articles

FIRST EDITION

beautybook

PREFACE

Here is a comprehensive list of the academic papers that I have perused.

– Ethan Lu
2024-07-06

Contents

I Research Notes For Articles 1

1 Vector Bundles with Semidefinite Curvature and Cohomology Vanishing Theorems 3

1.1 Preliminaries . . . 3
1.1.1 Correlation Techniques . . . 4
1.2 The Kähler Case . . . 4
1.2.1 An application of semidefinite theorem in Kähler case . . . 7
1.3 General case . . . 8
1.4 On vector bundles . . . 12
1.4.1 Terminologies . . . 12
1.5 Vanishing theorems for positive semidefinite vector bundles of rank k . . . 13
1.5.1 Kähler case over semidefinite vector bundles . . . 13
1.5.2 General case over semidefinite vector bundles . . . 14
1.6 The semi-curvature of a line bundle $L(E)$ over $P(E)$ . . . 14
1.6.1 Computation of $\Theta \wedge \varphi = 0$ . . . 16
1.6.2 Computation of the quadratic form $\Theta(\zeta^0, \eta)$ . . . 17
1.7 Summary and Reflection . . . 20

2 Logarithmic vanishing theorems on compact Kähler manifolds I 21

2.1 Terminologies . . . 21
2.2 Some problems and their solutions . . . 22
2.3 Preliminaries . . . 22
2.3.1 Proof of Theorem1.1 . . . 22

3 L^2-approach to the saito vanishing theorem 32

3.1 Terminologies . . . 32
3.2 Dolbeault Resolution for the de Rham Complex . . . 34
3.2.1 The Dolbeault resolution . . . 34
3.2.2 The logarithmic Holomorphic de Rham Resolution . . . 34
3.3 L^2-existence results . . . 35
3.4 Construction of the Kähler metric on the Complement . . . 37
3.4.1 Poincaré type Kähler metric on complement $X\backslash D$ . . . 38
3.4.2 Check the positivity of the commutator operator $[i\Theta_{\mathcal{L}\otimes B}, \Lambda_\omega]$ . . . 38
3.5 L^2–Dolbeault resolution . . . 39
3.5.1 L^2–Dolbeault resolution of the de Rham complex when there is a variation of Hodge structures on the complement of an SNC devisor . . . 39

3.6 Proof of the Main theorem 40

4 Logarithimic Vanishing Theorems for Effective q-Ample Divisors 42

4.1 Introduction 42

5 Notes for some new topics 44

5.1 Perverse Sheaf and Intersection Cohomology 44
5.1.1 Poincaré Duality 44
5.1.2 Understanding Why the Odd Betti Numbers of a Complex Projective Manifold are Even? 45
5.1.3 Lefschetz Hyperplane Section Theorem 46

Bibliography 47

Part I

Research Notes For Articles

This notes are mainly about some vanishing theorems and their proofs.

1 Vector Bundles with Semidefinite Curvature and Cohomology Vanishing Theorems

Part I

Sec 1.1 Preliminaries

Theorem 1.1.1. (*Kodaira vanishing theorem[7, P196]*)

Let $\mathcal{L}$ be a positive (ample) line bundle on a compact Kähler manifold X with $\dim X = n$. Then

$$H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0, \quad \textit{for } p + q > n. \tag{1.1}$$

Definition 1.1.1. (*Chern classes[7, P196]*)

Let $\{\widetilde{P}_k\}$ be the homogeneous polynomials with $\deg(\widetilde{P}_k) = k$ defined by

$$\det(\mathrm{Id} + B) = 1 + \widetilde{P}_1(B) + \ldots + \widetilde{P}_r(B).$$

Clearly, these $\widetilde{P}_k$ are *invariant*.
The Chern classes of a complex manifold X are

$$c_k(X) := c_k(\mathcal{T}_X) \in H^{2k}(X, \mathbb{R}),$$

where $\mathcal{T}_X$ is the holomorphic tangent bundle.

Definition 1.1.2. (*Chern forms[7, P196]*)

The *Chern forms* of a vector bundle E of rank k endowed with a connection ∇ are

$$c_k(E, \nabla) := \widetilde{P}_k\left(\frac{i}{2\pi} F_\nabla\right) \in \mathcal{A}_C^{2k}(M).$$

The *k-th Chern class* of the vector bundle E is induced cohomology class

$$c_k(E) := [c_k(E, \nabla)] \in H^{2k}(M, \mathbb{C}).$$

Theorem 1.1.2. (*Another description of Kodaira vanishing theorem[4, Introduction]*)

Let F be a holomorphic line bundle over a compact Kähler manifold M. If the Chern class $\mathcal{C}_\mathbb{R}(F) \in H^2(M, \mathbb{R})$ contains a negative definite form $\mathcal{X}(\mathcal{X} < 0)$ then all cohomology groups

$$H^q(M, \Omega(F)) = 0$$

when $q \leqslant n - 1$.

Theorem 1.1.3. (*Akizuki-Nakano Vanishing Theorem[4, Introduction]*)

(The generalization of Kodaira vanishing theorem by Akizuki-Nakano)
If $\mathcal{X} < 0$, then

$$H^q(M, \Omega^p(F)) = 0$$

when $p + q \leqslant n - 1$.

Theorem 1.1.4. (*Vesentini vanishing theorem[4, Introduction]*)

If $\mathcal{X}$ is semi-definite of rank, k (i.e. $\mathcal{X} \leqslant 0$ and $\mathcal{X}$ has k negative eigenvalues at each point of M) then

$$H^q(M, \Omega(F)) = 0 \qquad \& \qquad H^0(M, \Omega^q(F)) = 0$$

when $q \leqslant k - 1$.

Problem 1.1.1. (*[4, Introduction]*)

If $\mathcal{X} \leqslant 0$ with rank k, then

$$H^q(M, \Omega^p(F)) = 0$$

when $p + q \leqslant k - 1$?

1.1.1 Correlation Techniques

Problems in holomorphic vector bundles can be often reduced to similar problems in line bundles, by means of constructing the projective bundle PE over M and the tautological line bundle LE^{-1}.

By using the technique, we shall generalize to vector bundles some of the results of the following two section.

Sec 1.2 The Kähler Case

The first result says that let M be a compact Kähler manifold and let F be a holomorphic line bundle over M. If $C_{\mathbb{R}}(F)$ contains a form $\mathcal{X}$ whose associated hermitian form is negative semidefinite of rank k at each point of M, then

$$H^t(M, \Omega^s(F)) = H^s(M, \Omega^t(F)) = 0, \quad \text{for} \quad s + t \leqslant k - 1.$$

The proof mainly depends on the *Akizuki-Nakano Inequality* that given any harmonic (p, q)-form φ with values in E, then

$$([\Lambda, \mathrm{ie}(\Theta)]\varphi, \varphi) \geqslant 0, \tag{1.2}$$

where $e(\Theta)$ denotes the exterior multiplication of the matrix of local $(1, 1)$-form with the column vector φ. And in [2], we have known that $e(\Theta) = (\partial_E \overline{\partial} + \overline{\partial}\partial_E)\varphi$. Here is a simplified version.

In the Kähler case, let $F \to M$ be a holomorphic line bundle over a compact Kähler manifold of dimension n. Take $(U; (z^1, \cdots, z^n))$ be a local coordinate system on M and

$$\varphi = \sum \varphi_{A\bar{B}} \, \mathrm{d}z^A \wedge \mathrm{d}\bar{z}^B,$$

where $A = (\alpha_1, \ldots, \alpha_p)$ $(\alpha_1 < \cdots < \alpha_p)$, $B = (\beta_1, \ldots, \beta_q)$ $(\beta_1 < \cdots < \beta_q)$.

Denoting by $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n$ the eigenvalues of the hermitian form associated to $\mathcal{X}$, for any $z \in U$, compatible coordinates centered at z can be chosen in U in such a way that the fundamental form ω of the Kähler metric is given at z by

$$\omega = i \sum \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}$$

and

$$\mathcal{X} \;= \frac{i}{2\pi}\Theta \;= \frac{i}{2\pi}\sum_\alpha X_\alpha \, \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}.$$

Note that the expression of ω above is the consequence of diagonalization. And one can easily obtain these two equations by using simultaneously diagonalization for ω and $\mathcal{X}$.

Then we compute the formula of Akizuki-Nakano Inequality by

$$\begin{aligned}
& ([\Lambda, \mathrm{ie}(\Theta)]\varphi)_{A\bar{B}}(z) \\
= \;& [\Lambda, \mathrm{ie}(\Theta)]_{A\bar{B}}(z) \cdot \varphi_{A\bar{B}}(z) \\
= \;& [\Lambda, 2\pi i \mathcal{X}]_{A\bar{B}}(z) \cdot \varphi_{A\bar{B}}(z) \\
= \;& -\left[\sum_\alpha X_\alpha \, \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}, \Lambda\right]_{A\bar{B}} (z) \cdot \varphi_{A\bar{B}}(z) \\
= \;& -\left[\left(\sum_\alpha X_\alpha \, \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}\right) \wedge (\Lambda \varphi_{A\bar{B}})\right](z) + \Lambda\left[\left(\sum_\alpha X_\alpha \, \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}\right) \wedge \varphi_{A\bar{B}}\right](z) \\
\overset{\star}{=} \;& -\sum_{\alpha \in A\cap B} X_\alpha(z)\varphi_{A\bar{B}}(z) + \sum_{\alpha \notin A \cup B} X_\alpha(z)\varphi_{A\bar{B}}(z),
\end{aligned}$$

where

$$\left[\left(\sum_\alpha X_\alpha \, \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}\right) \wedge (\Lambda \varphi_{A\bar{B}})\right] = \begin{cases} \sum X_\alpha \varphi_{A\bar{B}}, & \alpha \in A \cap B, \\ 0, & \alpha \notin A \cap B, \end{cases}$$

and

$$\Lambda\left[\left(\sum_\alpha X_\alpha \, \mathrm{d}z^\alpha \wedge \overline{\mathrm{d}z^\alpha}\right) \wedge \varphi_{A\bar{B}}\right] = \begin{cases} \sum X_\alpha \varphi_{A\bar{B}}, & \alpha \notin A \cup B, \\ 0, & \alpha \in A \cup B. \end{cases}$$

Thus, if the following relations are satisfied at each point $z \in M$

$$(X_{t_1} + \cdots + X_{t_s}) - (X_{j_1} + \cdots + X_{j_{n-t}}) > 0$$

for each choice of $i_1 < \cdots < i_s$ and $j_1 < \cdots j_{n-t}$ and for all $s \leqslant t$, then

$$([\Lambda, \mathrm{ie}(\Theta)]\varphi, \varphi)(z) \leqslant 0. \tag{1.3}$$

Definition 1.2.3. ***(Positive Hermitian line bundle)***

A hermitian holomorphic line bundle E on X is said to be positive (negative/semi-positive/semi-negative) if the hermitian matrix (Component) of its Chern curvature form

$$i\Theta(E) = i \sum_{1 \leqslant j,k \leqslant n} c_{jk}(z) \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

is positive (negative/semi-positive/semi-negative) definite at every point $z \in X$.

In[3, P334], by Prop VI-8.3, we gain that

$$\begin{aligned}\langle [i\Theta(E), \Lambda]u, u\rangle &= \sum_{J,K} (\sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leqslant j \leqslant n} \gamma_j) |u_{J,K}|^2 \\ &\geqslant (\gamma_1 + \cdots + \gamma_q - \gamma_{p+1} - \cdots - \gamma_n)|u_{J,K}|^2\end{aligned}$$

for any form $u = \sum_{J,K} u_{J,K} \zeta_J \wedge \bar{\zeta}_K \in \Lambda^{p,q} T^*(X)$. Then we have

$$\begin{aligned}([\Lambda, \mathrm{ie}(\Theta)]\varphi, \varphi)(z) &= -\langle [i\Theta(E), \Lambda]u, u\rangle \\ &\leqslant (\gamma_1 + \cdots + \gamma_q - \gamma_{p+1} - \cdots - \gamma_n)|u_{J,K}|^2 \\ &\leqslant \left((X_{j_1} + \cdots + X_{j_{n-t}}) - (X_{i_1} + \cdots + X_{i_s})\right) |u_{J,K}|^2 \\ &\leqslant 0.\end{aligned}$$

By (1.3), consequently, in view of inequality (1.2), any harmonic (s,t)-form vanishes identically, i.e. the Lemma 1.1 of the paper.

Remark. *By (1.2) and (1.3), we have [2, Lemma2, P483]*

$$([\Lambda, \mathrm{ie}(\Theta)]\varphi, \varphi)(z) = 0 \iff \varphi \in H'^{p,q}(M,F) \cap H''^{p,q}(M,F)$$

As $H'^{p,q}(M,F) \cong H^{p,q}_{\bar\partial}(M,F)$ *and* $H'^{p,q}(M,F) \cong H^{p,q}_{\partial}(M,F)$*, one has*

$$H'^{p,q}(M,F) \cap H''^{p,q}(M,F) \cong H^{p,q}_{\bar\partial}(M,F) \cap H^{p,q}_{\partial}(M,F) = \{0\}. \tag{1.4}$$

Thus $\varphi \in \{0\} \Longrightarrow \varphi \equiv 0$*, i.e.* ***any harmonic*** (p,q)***-form*** φ ***vanishes identically***.

Suppose $\mathcal{X} \leqslant 0$ of rank k. Now, for any $\mu \in \mathbb{R}^+$, the form $(\omega - \mu\mathcal{X})$ is still a Kähler form on M. Denoting the eigenvalues of $\mathcal{X}$ with respect to the metric induced by ω as before, the eigenvalues of $\mathcal{X}$ with respect to the metric induced by $(\omega - \mu\mathcal{X})$ are

$$\frac{X_i}{(1-\mu X_i)}, \quad i = 1, \ldots, n.$$

Now, if μ is sufficient large, we have

$$\sum_{i=1}^{s} \frac{X_i}{(1-\mu X_i)} > \sum_{j=t+1}^{n} \frac{X_j}{(1-\mu X_j)}. \tag{1.5}$$

The computation should be the following.

Firstly, we gain the first inequality

$$\sum_{i=1}^{s} \frac{(X_i - X_{t+i})}{(1-\mu X_i)(1-\mu X_{t+i})} \geqslant \frac{s(X_1 - X_k)}{(1-\mu X_1)(1-\mu X_k)} = s\left(\frac{X_1}{(1-\mu X_1)} - \frac{X_k}{(1-\mu X_k)}\right).$$

As $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n$, then

$$\frac{X_i}{(1-\mu X_i)} \leqslant \frac{X_j}{(1-\mu X_j)}, \quad \text{where} \quad 1 \leqslant i \leqslant j \leqslant n.$$

But for the associated hermitian form Θ is negative semidefinite of rank k, there are k negative eigenvalues of $\mathcal{X}$, which can be written as $X_1 \leqslant \cdots \leqslant X_k < 0 \leqslant X_{k+1} \leqslant \cdots \leqslant X_n$ without loosing generality. $(X_l \geqslant 0, l = k+1, \ldots, n)$

Thus, we have

$$\begin{aligned}\sum_{i=1}^{s} \frac{(X_i - X_{t+i})}{(1-\mu X_i)(1-\mu X_{t+i})} &= \sum_{i=1}^{s} \left[\frac{X_i}{(1-\mu X_i)} \downarrow - \frac{X_{t+i}}{(1-\mu X_{t+i})} \uparrow\right] \\ &\geqslant s\left(\frac{X_1}{(1-\mu X_1)} - \frac{X_k}{(1-\mu X_k)}\right).\end{aligned}$$

Secondly, we abtain the second inequality

$$\begin{aligned}\sum_{j=s+t+1}^{k} \frac{X_j}{(1-\mu X_j)} &= \sum_{j=s+t+1}^{n} \frac{X_j}{(1-\mu X_j)} \\ &= \sum_{j=s+1}^{n-t} \frac{X_{j+t}}{(1-\mu X_{j+t})} \leqslant [k-(s+t+1)]\frac{X_k}{(1-\mu X_k)}\end{aligned}$$

by

$$\frac{X_j}{(1-\mu X_j)} \leqslant \frac{X_k}{(1-\mu X_k)}, \quad \text{where } X_1 \leqslant \cdots \leqslant X_k \leqslant 0 \leqslant \underbrace{X_{k+1} \leqslant \cdots \leqslant X_n}_{n-k}.$$

Then, for each $z \in M$, we can find $\bar{\mu} \in \mathbb{R}^+$ so that for any $\mu \geqslant \bar{\mu}$ the hermitian form associated to Θ satisfies condition of Lemma 1.1 for any pair (s,t) such that $s+t \leqslant k-1$, with respect to the Kähler metric $(\omega - \mu\mathcal{X})$. ***This means that for any Kähler form (Not just specific one), by lemma 1.1 of paper, any harmonic*** (s,t)***-form vanishes identically.*** Eventually, we gain the Theorem 2.1, i.e., the result we have written in the beginning of this section.

1.2.1 An application of semidefinite theorem in Kähler case

Theorem 1.2.5. (*Lefschetz theorem on hyperplane sections*)

If the Chern class $C_{\mathbb{R}}([S]) \in H^2(M,\mathbb{R})$ of $[S]$ contains a form $\mathcal{X} \geqslant 0$ of rank k, then

$$\rho_S^* \text{ is an isomorphism for } s \leqslant k-2$$

and

$$\rho_S^* \text{ is injective for } s = k-1.$$

***Proof*.** The main body remains the same as [1], and now we provide the kernel of the proof. We gain the **exact cohomology sequence** from above

$$\cdots \to H^q(M, \Omega^p([S]^{-1})) \to H^q(M, \Omega'^p) \to H^q(S, \Omega^{p-1}([S]^{-1}|_S)) \to \cdots.$$

Applying the vanishing theorem in Kahler case, we have

$$H^q\left(M, \Omega'^p\right) = 0, \quad \text{whenever} \quad p+q \leqslant k-1.$$

The exact Cohomology sequence yields the **isomorphisms**

$$H^q(M, \Omega^q) \cong H^q(S, \Omega^p), \quad \text{for} \quad p+q \leqslant k-2 = (k-1)-1 \tag{1.6}$$

and the **injectivity** of the map

$$H^q(M, \Omega^p) \to H^q(S, \Omega^p), \quad \text{whenever} \quad p+q = k-2.$$

Since

$$H^s(M, \mathbb{C}) \cong \oplus_{p+q=s} H^q(M, \Omega^p)$$

and a similar decomposition holds for S, the conclusion follows. ■

Sec 1.3 General case

In the general case, the proof is very different, which is mainly depends on the *local expression of the Laplace-Beltrami operator and on a result about hermitian forms*, due to E. Calabi and *Lectures on Convexity of Complex manifolds and Cohomology vanishing theorems* by E. Vesentini.

If the hermitian metric on X is a Kähler metric [10, P72], then

$$\overline{\partial} = \tilde{\partial}, \quad \vartheta = \tilde{\vartheta}, \quad \square = \tilde{\square}.$$

For $\varphi \in C^{pq}(X, E)$

$$(\tilde{\vartheta}\varphi)^a_{AB'} = (-1)^{p-1} \nabla_\alpha \varphi^a_A \alpha_{\overline{B'}},$$

so that, exactly as in the case of the Laplacian $\triangle$ in Chapter 2, we have

$$(\tilde{\square}\varphi)^a_{\overline{AB}} = -\nabla_\alpha \nabla^\alpha \varphi \frac{a}{AB} + \sum_{r=1}^{q} (-1)^{r-1} \left(\nabla_\alpha \nabla_{\overline{\beta_r}} - \nabla_{\overline{\beta_r}} \nabla_\alpha\right) \varphi^a_A \alpha_{\overline{B'_r}} \tag{1.7}$$

where

$$\nabla^\alpha = g^{\alpha\bar{\beta}} \nabla_{\bar{\beta}},$$

and $\quad A = (\alpha_1, \ldots \alpha_p)\,, B = (\beta_1, \ldots \beta_q)\,, B'_r = \left(\beta_1, \ldots, \hat{\beta}_r, \ldots, \beta_q\right).$

In view of the Ricci identity, the summand of (1.7) can be expressed by

$$\sum_{r=1}^{q} (-1)^{r-1} \left(\nabla_\alpha \nabla_{\overline{\beta_r}} - \nabla_{\overline{\beta_r}} \nabla_\alpha\right) \varphi^a_A \alpha_{\overline{B'_r}} = (\tilde{\mathcal{K}}\varphi)^a AB \tag{1.8}$$

where $\tilde{\mathcal{K}}$ is a mapping

$$\tilde{\mathcal{K}} : C^{pq}(X, E) \to C^{pq}(X, E),$$

which is linear over C^∞ functions, whose local expression involves linearly (with integral coefficients) only the coefficients of the curvature forms, s and L, of E and $\Theta_\circ$.

By [10, Remark (3) after Lemma 3.2], we have

$$\left(\tilde{\mathcal{K}}_\varphi\right)^a_{A\bar{B}} = \sum_{r=1}^{q} (-1)^r s^a_{b\bar{\beta}_r\alpha} \varphi^b_A \alpha_{\overline{B'_r}} + \left(\tilde{\mathcal{K}}_\circ \varphi\right)^a_{A\bar{B}}, \tag{1.9}$$

where $\tilde{\mathcal{K}}_\circ$ involves only the curvature tensor of $\Theta_\circ$, and is completely independent of E.

Formula (1.7) can be also written as

$$(\tilde{\Box}\varphi)^a_{A\bar{B}} = -\nabla_\alpha \nabla^\alpha \varphi^a_{A\bar{B}} + (\tilde{\mathcal{K}}\varphi)^a_{A\bar{B}}. \tag{1.10}$$

Now

$$\begin{aligned}\Box &= (\bar{\partial}\vartheta + \vartheta\bar{\partial}) = (\bar{\partial} + S)(\tilde{\vartheta} + T) + (\tilde{\vartheta} + T)(\tilde{\partial} + S) \\ &= \tilde{\Box} + \tilde{\partial}T + T\tilde{\partial} + \tilde{\vartheta}S + S\tilde{\vartheta} + ST + TS.\end{aligned}$$

It follows that

Lemma 1.3.1. ***([10, Lemma 3.4.])***

For any $\varphi \in C^{pq}(X, E)$

$$(\Box\varphi)^a_{A\bar{B}} = (\tilde{\Box}\varphi)^a_{A\bar{B}} + ((F_1\varphi)^a_{A\bar{B}} + (F_2\nabla'\varphi)^a_{A\bar{B}} + (F_3\nabla''\varphi)^a_{A\bar{B}}) \tag{1.11}$$

where

$$\begin{aligned}F_1 &: C^{pq}(X, E) \to C^{pq}(X, E),\\ F_2 &: C^{pq}(X, E \otimes \Theta^*_o) \to C^{pq}(X, E),\\ F_3 &: C^{pq}(X, E \otimes \Theta^*_o) \to C^{pq}(X, E),\end{aligned}$$

are linear over C^∞ functions. Their local expression involves the tensor product and its first derivatives.

Then the Laplace-Beltrami operator $\Box$ on $(0, q)$-forms with coefficients in $F \otimes D$ is given locally by

$$(\Box\varphi)_{\overline{B}} = -\nabla_\alpha \nabla^\alpha \varphi_{\overline{B}} + (\mathcal{K}\varphi)_{\overline{B}} + \sum g^{\bar{\alpha}\beta}\Gamma^{\overline{\lambda}}_{\bar{\alpha}\beta}\nabla_{\overline{\lambda}}\varphi_{\overline{B}}, \tag{1.12}$$

where B is a set of q indices $B = (\beta_1, \cdots, \beta_q)$ and $\Gamma^{\overline{\lambda}}_{\bar{\alpha}\beta}$ are components of the Riemann-Christoffel connection defined by the hermitian metric on M and $\mathcal{K}$ is a linear operator on $(0, q)$-forms which splits as the sum of $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_{\mathcal{X}}$.

Lemma 1.3.2. ***([10, Lemma 2.2 by E. Calabi, P80])***

Let H be a hermitian quadratic differential form on X and G a hermitian metric on X. Assume that H has at least p positive eigen values. Let $\varepsilon_1(x), \ldots, \varepsilon_n(x)$ be the eigen values of H (w.r.t. G) at x in decreasing order: $\varepsilon_r(x) \geq \varepsilon_{r+1}(x)$ Then given $c_1, c_2 > 0, G$ can be so chosen that

$$l_H(x) = c_1\varepsilon_p(x) + c_2 \operatorname{Inf}(0, \varepsilon_n(x)) > 0, \quad \textit{for all } x \in X.$$

Proof. Let G be any complete hermitian metric whatever. Let $\sigma_1(x), \ldots, \sigma_n(x)$ be the eigen-values of H with reference to G arranged in decreasing order. We construct now a metric G on X whose eigen values are functions of $\{\sigma_i(x)\}_{1\leq i\leq n}$ as follows: let $\lambda : X \to \mathbb{R}$ be a $C^\infty-$ function (we will impose conditions on λ letter); let U be a coordinate open set in X with holomorphic coordinates $(z^1, \ldots, z^n)$; in U, we have $G = G_{U\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ so that $(G_{U\alpha\bar{\beta}})_{\alpha\beta}$ is a function whose values are positive definite hermitian matrices; then the matrix valued function $\widehat{G}_U = \left(\widehat{G}_{U\alpha\bar{\beta}}\right)_{\alpha\beta}$ where $\widehat{G} = \sum \widehat{G}_{U\alpha\bar{\beta}}dz^\alpha dz^\beta$ in U is defined by

$$\widehat{G}_U^{-1} = G_U^{-1}\sum_{r=0}^{\infty}\frac{\lambda(x)^r}{(r+1)!}\left(H_U G_U^{-1}\right)^r$$

where H_U is the matrix valued function $(H_{U\alpha\beta})$ defined by

$$H = \sum H_{U\alpha\bar{\beta}} dz^\alpha dz^\beta$$

in U.

We now assert that $\widehat{G}_U$ define a global hermitian defferential form on X and under a suitable choice of λ, it is positive definite. To see that $\widehat{G}_U$ defines a global hermitian differential form on X, we need only prove the following. Let V be another coordinate open set with coordinates complex $(w^1, \ldots, w^n)$. Let $J = \frac{\partial(z^1,\ldots,z^n)}{\partial(w^1,\ldots,w^n)}$ be the Jacobian matrix. As before let $G_V = (G_{V\alpha\beta})$ be defined by $G = \sum G_{V\alpha\bar{\beta}} dw^\alpha d\bar{w}^\beta$ in V. Then if $\widehat{G}_V$ is defined starting from G_V as $\widehat{G}_U$ from G_U, we have

$$J\widehat{G}_U^t \bar{J} = \widehat{G_V}$$

We have in fact, writing J^* for ${}^t\bar{J}^{-1}$,

$$J^*\widehat{G}_U^{-1}J^{-1} = G_V^{-1}J\left(\sum_{r=0}^{\infty}\frac{\lambda(x)^1}{(r+1)!}\left(H_U G_U^{-1}\right)^r\right)J^{-1}, \text{since } JG_U^t\bar{J} = G_V.$$

It follows from the above that

$$\begin{aligned} J^*\widehat{G}_U^{-1}J^{-1} &= G_V^{-1}\sum_{r=0}^{\infty}\frac{\lambda(x)^r}{(r+1)!}\left(JH_U G_U^{-1}J^{-1}\right)^r \\ &= G_V^{-1}\sum_{r=0}^{\infty}\frac{\lambda(x)^r}{(r=1)!}\left(H_V J^* G_U^{-1}J^{-1}\right)^r \end{aligned}$$

since $JH_U^t\bar{J}H_V$. Hence we obtain

$$\begin{aligned} J^*\widehat{G}_U^{-1}J^{-1} &= G_V^{-1}\sum_{r=0}\frac{(\lambda(x))^r}{(r+1)!}\left(H_V G_V^{-1}\right)^r \\ &= \widehat{G}_V^{-1} \end{aligned}$$

This proves that $\widehat{G}_U$ defines on X a global hermitian differential. We next show that $\widehat{G}$ is positive definite. For this we look for the eigenvalues of $\widehat{G}$ with reference to G. TO compute these, we may assume, in the above formula for $\widehat{G}_U$, that G_U is the identity matrix Then we have

$$\widehat{G}_U = \sum_{r=0}^{+\infty}\frac{\lambda(x)^r}{(r+1)!}H_U^r$$

It follows that the eigen values of $\widehat{G}_U$ are

$$\left\{\sum_{r=0}^{+\infty}\frac{\lambda(x)^r}{(r+1)!}\sigma_q(x)^r\right\}_{1\le q\le n}.$$

It is easily seen that these are all strictly greater than zero: this assertion simply means this: $f(t) = \frac{e^t-1}{t} = \sum_{r=0}^{\infty}\frac{t^r}{(r+1)!}$ for $t \neq 0$, $f(0) = 1$ (which is continuous in t) is everywhere greater than 0 .

We will now look for conditions on λ such that $\widehat{G}$ satisfies our requirements. From the formula for $\widehat{G}$, we have

$$H_U\widehat{G}_U^{-1} = \sum_{r=0}^{+\infty}\frac{\lambda(x)^r}{(r+1)}\left(H_U G_U^{-1}\right)^{r+1}.$$

Now the eigenvalues of H with respect to $\widehat{G}$ (resp. G) are simply those of the matrix $H_U\widehat{G}_U^{-1}$ (resp. $H_U G_U^{-1}$), Hence these eigenvalues $\varepsilon_q(X)$ of H with reference to $\widehat{G}$ are

$$f\left(\lambda(x), \sigma_q(x)\right)$$

where $f(s,t)$ is the function on $\mathbb{R}^2$ defined by

$$f(s,t) = \sum_{r=0}^{t\infty} \frac{S^r}{(r+1)} t^{r+1}$$

Since $\frac{\partial f(s,r)}{\partial t} = e^{st} > 0$ for any, $f(s,t)$ is monotone increasing in t. Hence we have

$$\varepsilon_r(x) \geqslant \varepsilon_{r+1}(x) \text{ for } 1 \leqslant r \leqslant n-1.$$

Moreover $f(s,t) \geqslant t$ for $s \geqslant 0$. Thus, if we choose $\lambda(x) \geqslant 0$ for every $x \in X$, then $\varepsilon_q(x) \geqslant \sigma_q(x) > 0$.

The choice of $\lambda(x)$ is now made as follows . Let, for every integer $v > 0.B_v = \{x \mid d\left(x, x_0\right) \leqslant \gamma\}$ for some $x_0 \in X$, the distance being i the metric G. The B_v are then compact. Let $b_v = \operatorname{Inf}_{x \in B_\gamma}\left(\sigma_p(x)\right)$.

Then $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{v+1} \geqslant \ldots$.

Let $b(x)$ be a C^∞ function on X such that $b(x) > 0$ for $x \in X$ and $b(x) < b_v$ in $B_v - B_{v-1}$. Then clearly $b(x) \leq \sigma_p(x)$.

Finally let $\rho(x)$ be a C^∞ function on X such that $\rho(x) \geqslant d\left(x, x_0\right)$, and $k > \sqrt{\frac{C_2}{C_1}} b_1$ be a real constant. Set $\lambda(x) = \frac{2ke^{\rho(x)}}{b^2(x)}$. We have then

$$\varepsilon_q(x) = f\left(\lambda(x), \sigma_p(x)\right) = \sigma_p(x) + \frac{\lambda(x)}{2!}\sigma_p(x)^2 + \cdots$$

so that

$$\varepsilon_p(x) \geqslant \frac{ke^{\rho(x)}}{b^2(x)}\sigma_p(x)^2 \geqslant ke^{\rho(x)} \geqslant k.$$

On the other hand,

$$\varepsilon_n(x) = f\left(\lambda(x), \sigma_n(x)\right) = \frac{1}{\lambda(x)}\left\{e^{\lambda(x)\sigma_n(x) - 1}\right\} \geqslant \frac{-1}{\lambda(x)} = -\frac{b^2(x)}{2ke^{\rho(x)}} \geqslant -\frac{b_1^2}{k}$$

$$C_1\varepsilon_p(x) + C_2 \operatorname{Inf}\left(0, \varepsilon_n(x)\right) \geqslant C_1 k - C_2\frac{b_1^2}{k} \geqslant \frac{1}{k}\left(C_1 k^2 - C_2 b_1^2\right) > 0.$$

■

The general version of the theorem will be described as follows :

Theorem 1.3.6. (*The general semidefinite vanishing theorem*)

Let M be a compact hermitian manifold. Let F be a holomorphic line bundle and D be a holomorphic vector bundle over M.
If $C_{\mathbb{R}}(F)$ contains a form $\mathcal{X}$ whose associated hermitian form has at least k positive eigenvalues at each ponit of M, then there exists a positive integer μ_0 such that

$$H^q(M, \Omega(F^\mu \otimes D)) = 0$$

for all $\mu \geqslant \mu_0$, and all $q \leqslant n-k+1$.

***Proof*.** By **Lemma 1.3.2**, M can be equipped with a new hermitian metric in such a way that, denoting by $X_1(z) \geqslant \cdots \geqslant X_n(z)$ the eigenvalues of $\mathcal{X}$ w.r.t. this metric, then $X_k(z) > 0$ and

$$X_k(z) + n \cdot \inf(0, X_n(z)) > 0, \quad \text{at each point } z \in M.$$

As $K = K_0 + K_{\mathcal{X}}$ is a linear operator, then $A(K\varphi, \varphi) = A(K_0\varphi, \varphi) + A(K_{\mathcal{X}}\varphi, \varphi)$. Aaccording to a straightforward computation, we abtain

$$A(K_{\mathcal{X}}\varphi, \varphi)(z) \geqslant (X_k(z) + n \cdot \inf(0, X_n(z)))A(\varphi, \varphi)(z)$$

by using the fact that for $q \geqslant n - k + 1$, there is at least one of the indices $\beta \leqslant k$.

And since M is compact, there exist a positive C s.t. $A(K_0\varphi, \varphi)(z) \geqslant -\mathrm{CA}(\varphi, \varphi)(z)$ at every point $z \in M$.(K_0 is a bounded linear operator on M when M is comapct.) Then we choose a μ_0 s.t.

$$\mu_0(X_k(z) + n \cdot \inf(0, X_n(z))) - (C + 1) > 0, \quad \text{at each point } z \in M,$$

which implies $A(K\varphi, \varphi) \geqslant A(\varphi, \varphi)$. (

$$\begin{aligned} A(K\varphi, \varphi)(z) &\geqslant [(X_k(z) + n \cdot \mathrm{ing}(0, X_n(z)) - C]A(\varphi, \varphi)(z) \\ &\geqslant \frac{[(1 - \mu_0)C + 1]}{\mu_0} \cdot A(\varphi, \varphi)(z) \geqslant A(\varphi, \varphi)(z) \end{aligned}$$

)

Thus, if φ is any harmonic $(0, q)$-form with coefficients in $F^{\mu} \otimes D$, with $\mu \geqslant \mu_0$ and $q \geqslant n - k + 1$, we have

$$0 \leqslant \|\varphi\|^2 = (\varphi, \varphi) \leqslant (K\varphi, \varphi) \leqslant 0,$$

which means that $\varphi \equiv 0$. ■

Sec 1.4 On vector bundles

1.4.1 Terminologies

Definition 1.4.4. (*Hermitian metric*)

On a holomorphic vector bundle with a hermitian metric h, there is a unique connection compatible with h and the complex structure. Namely, it must be $\nabla = \partial + \bar{\partial}$, where $\partial s = h^{-1}\partial\, \mathrm{hs}$.

Table 1.1: Terminologies Interpretation

Terminologies	Interpretations
PE	$= (E - 0)/\mathbb{C}^*$,
$(E - 0)$	The bundle space E minus its zero section,
Terminologies	***Interpretations***

Continued on next page

Table 1.1: Terminologies Interpretation (Continued)

Terminologies	Interpretations
Curvature form $\hat{\Theta}$	$\hat{\Theta} = \sum_{i,j} \Theta^{r}_{\mathrm{ri}\bar{j}} \, \mathrm{dz}^i \wedge \overline{\mathrm{dz}}^j - \sum_1^{r-1} d\zeta^\alpha \wedge \overline{d\zeta^\alpha}$,
D	A holomorphic vector bundle over M,
$S^k E$	k-th symmetric tensor power of E,
LE	the associated complex line bundle over PE.
Terminologies	*Interpretations*

We have the following homeomorphisms [9, Theorem 2.1, P504 and P502]:

$$H^q(\mathrm{PE}^*, \Omega((\mathrm{LE}^*)^{-k} \otimes \pi^* D)) \cong H^q(M, \Omega(S^k E \otimes D))$$

and

$$H^q(\mathrm{PE}^*, \Omega^p(\mathrm{LE}^*)^{-1}) \cong H^q(M, \Omega^p(E)).$$

Sec 1.5 Vanishing theorems for positive semidefinite vector bundles of rank k

In order to define the following concept, it is necessary to calculate the Hermitian quadratic form $\Theta(\zeta, \eta)$, which is strongly associated with the definition. (cf [8],proof of Proposition 6.3)

Definition 1.5.5. (*Vector bundle being positive-semidefinite of rank k*)

E is said to be *positive semidefinite of rank k* if there exists a hermitian metric on E whose curvature tensor Θ satisfies the following condition: for all $\zeta \in \mathbb{C}^r - 0$, the quadratic form on the variable η : $\Theta(\zeta, \eta)$ *is positive semidefinite of rank k at each point of M*, where

$$\Theta(\zeta, \eta) = \sum_{i,j} \Theta^{\rho}_{\sigma i \bar{j}} \zeta^\sigma \bar{\zeta}^\rho \eta^i \bar{\eta}^j.$$

1.5.1 Kähler case over semidefinite vector bundles

Theorem 1.5.7. (*Semidefinite vanishing theorem for vector bundle of rank k (Kähler Case)*)

Let E be positive semidefinite (or negative semidefinite) of rank k at each point z of a compact Kähler manifold M. Then

$$H^q(M, \Omega^p(E)) = 0 \quad \text{if} \quad p + q \geqslant 2n - (k - r).$$

(respectively, $H^q(M, \Omega^p(E)) = 0$ if $p + q \leqslant k - r$. (By Serre Duality Theorem.))

Theorem 1.5.8. (*The generalization of Theorem 3.1)*

$$\begin{cases} \rho_S^* \text{ is an isomorphism}, & \text{if } s \leqslant k-r-2, \\ \rho_S^* \text{ is injective}, & \text{if } s = k-r-1. \end{cases}$$

This paper gives an application of its vanishing theorem of Kähler case over semidefinite vector bundle, which is the *Lefschetz theorem on hyerplane sections*. Here, S will be a non-singular complex submanifold of codimension r regularly imbedded in the compact Kähler manifold M, and S will be assumed to be the zero set of a holomorphic section ζ of a holomorphic r-vector bundle $E \to M$. Let $\rho_S^* : H^s(M, \mathbb{C}) \to H^s(S, \mathbb{C})$. Let $\widehat{\zeta} \in H^0(PE^*, (LE^*)^{-1})$ be the holomorphic section of the line bundle $(LE^*)^{-1}$ corresponding to ζ. Thus, $\widehat{\zeta} = \{p \colon \widehat{\zeta}(p) = 0\}$ is a non-singular submanifold of codimension 1 in PE^*.

1.5.2 General case over semidefinite vector bundles

Theorem 1.5.9. (*Semidefinite vanishing theorem for vector bundle of rank k (General Case))*

Let M be a compact complex manifold, and assume that there exists a hermitian metric on E, whose curvature tensor Θ satisfies the following condition at each point of M.
For any $\zeta \in \mathbb{C}^r - 0$, the quadratic form in η, $\Theta(\zeta, \eta)$, has at least $n - k_1$ positive eigenvalues and at least k_2 negative eigenvalues. Then

$$H^q(M, \Omega(S^\mu E \otimes D)) = 0$$

for any $q \notin (k_1, \ldots, k_2)$ if $\mu \geqslant 0$.

Sec 1.6 The semi-curvature of a line bundle $L(E)$ over $P(E)$

Proposition 1.6.1. (PROPOSITION 6.1. OF [9])

Given a point $o \in M$, there exist local holomorphic sections $s_1, \cdots, s_r$ around o such that

$$h_{\alpha\bar{\beta}} = \delta_{\alpha\beta} \quad \textit{and} \quad dh_{\alpha\bar{\beta}} = 0 \quad \textit{at } o.$$

Proof. Choose local holomorphic sections $t_1, \cdots, t_r$ around o which are orthonormal at o. We set

$$s_\alpha = \sum a_\alpha^\beta t_\beta \quad (a_\alpha^\beta : \text{ holomorphic })$$

and try to find (a_α^β) such that $a_\alpha^\beta = \delta_\alpha^\beta$ at o and $s_1, \cdots, s_r$ satisfy the required second condition. If we set

$$g_{\alpha\bar{\beta}} = h\left(t_\alpha, \bar{t}_\beta\right),$$

then

$$h_{\alpha\bar{\beta}} = h\left(s_\alpha, \bar{s}_\beta\right) = \sum a_\alpha^\gamma g_{r\bar{\delta}} \bar{a}_\beta^{\delta},$$

or in matrix form

$$H = {}^tA \cdot G \cdot \bar{A}.$$

We want to find A such that $A = I$ at o and $dH = 0$ at o. Since

$$\partial H = \partial^t A \cdot G \cdot \bar{A} + {}^tA \cdot \partial G \cdot \bar{A},$$

it suffices to set

$$a_\alpha^\beta = \delta_\alpha^\beta - \Sigma \left(\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^j} \right)_0 \cdot z^j,$$

where $z^1, \cdots, z^n$ is a local coordinate system with origin o. ■

Proposition 1.6.2. (*Proposition 6.3. of [9]*)

If E is a hermitian vector bundle with negative curvature (resp. semi-negative curvature), then the line bundle $L(E)$ over $P(E)$ with the induced hermitian metric has negative (resp. semi-negative) curvature.

***Proof*.** The naturally induced hermitian metric $\tilde{h}$ in $L(E)$ may be described as follows. Since $L(E)$ minus its zero section is naturally isomorphic to E minus its zero section

$$(L(E) - 0) \cong (E - 0),$$

every nonzero element X of $L(E)$ may be identified with an element of E, and

$$\tilde{h}(X, X) = h(X, X).$$

Fixing a point o in the base manifold M, we choose holomorphic sections $s_1, \cdots, s_r$ in a neighborhood of o with the properties stated in Proposition 6.1. Then we may write

$$h(X, X) = \Sigma h_{\alpha\bar{\beta}} \xi^\alpha \bar{\xi}^\beta \quad \text{for } X = \Sigma \xi^\alpha s_\alpha.$$

We shall *compute the Ricci tensor of the line bundle $L(E)$ at an arbitrarily fixed point of $P(E)$ which lies over $o \in M$*. This point is represented by a unit vector $X_0 \in E$. Applying a unitary transformation to $s_1, \cdots, s_r$, we may assume that $X_0 = s_r(0)$. We take $z^1, \cdots, z^n, \xi^1, \cdots, \xi^{r-1}$ as a local coordinate system around $[X_0]$ in $P(E)$, $[X_0]$ denotes the point of $P(E)$ represented by X_0. Then the components of the Ricci tensor of $L(E)$ at $[X_0]$ are given by

$$\begin{pmatrix} -\dfrac{\partial^2 \log h(X,X)}{\partial z^i \partial \bar{z}^j} & -\dfrac{\partial^2 \log h(X,X)}{\partial z^i \partial \bar{\xi}^\beta} \\ -\dfrac{\partial^2 \log h(X,X)}{\partial \xi^\alpha \partial \bar{z}^j} & -\dfrac{\partial^2 \log h(X,X)}{\partial \xi^\alpha \partial \bar{\xi}^\beta} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial^2 \log h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} & 0 \\ 0 & -\delta_\beta^\alpha \end{pmatrix}$$

where $i, j = 1, \cdots, n$ and $\alpha, \beta = 1, \cdots, r-1$. It is clear that this matrix is negative (semi-) definite if the curvature of E is (semi-) negative. ■

Remark. *If E has (semi-) positive curvature, its dual E^* has (semi-) negative curvature by Proposition 6.2 and hence the line bundle $L(E*)$ over $P(E^*)$ has (semi-) negative curvature by Proposition 6.3 and its dual $L(E^*)^{-1} = L(E^*)^*$ has (semi-) positive curvature. But $L(E)$ itself does not have (semi-) positive curvature.*

From *Proposition 6.3*, we obtain immediately the following

Theorem 1.6.10. (TEOREM 6.4. OF [9])

A hermitian vector bundle E with negative (resp. semi-negative, positive, or semi-positive) curvature is negative (resp. semi-negative, positive, or semi-positive).

We do not know if the converse is true, e.g., if a negative vector bundle E admits a hermitian metric with negative curvature. For a line bundle E, by definition E is negative (resp. positive) if and only if it admits a hermitian metric with negative (resp. positive) curvature. It is, however, not clear if a semi-negative (resp. semi-positive) line bundle admits a hermitian metric with semi-negative (resp. semi-positive) curvature.

So form the proof of **Proposition 1.6.2** we get the curvature of $L(E)$ with respect to the induced hermitian metric $\tilde{h}$ at the point $[X_0] \in P(E)$

$$\begin{aligned}\Theta_{L(E)}([X_0]) &= \sum_{i,j}\left(-\frac{\partial^2 \log h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j}\right) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j - \sum_{\alpha=1}^{r-1} \mathrm{d}\xi^\alpha \wedge \mathrm{d}\bar{\xi}^\alpha. \\ &= \sum_{i,j} \Theta^r_{ri\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j - \sum_{\alpha=1}^{r-1} \mathrm{d}\xi^\alpha \wedge \mathrm{d}\bar{\xi}^\alpha.\end{aligned}$$

There is still an issue.

$$\Theta_{i\bar{j}} = \sum \Theta^\alpha_{\alpha i\bar{j}} = -\frac{\partial^2 \log \det h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} ? =? -\frac{\partial^2 \sum_{\alpha,\beta} \log h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} = \sum_{\alpha,\beta}\left(-\frac{\partial^2 \log h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j}\right).$$

1.6.1 Computation of $\Theta \wedge \varphi = 0$

For P.52 of [4]

It can be verified that using (1), if E is negative semidefinite, then any harmonic $(p, 0)$- or $(0, p)$- form with coefficients in E has to satisfy the condition : $\Theta \wedge \varphi = 0$ at each point of M.

In fact, let φ be a harmonic $(p, 0)$- or $(0, p)$- form with coefficients in E, then clearly, we have $\Lambda\varphi \equiv 0$, which shows that (1) can be written as $(\sqrt{-1}\Lambda e(\Theta)\varphi, \varphi) \geqslant 0$. And the pointwise scalar product $A(\sqrt{-1}\Lambda e(\Theta)\varphi, \varphi) \leqslant 0$ at each point of M. Thus by (1) it must be $(\Lambda e(\Theta)\varphi, \varphi) = 0$ and therefore, for (2), we have

$$\begin{aligned}&(\Lambda(\partial_E\bar{\partial} + \bar{\partial}\partial_E)\varphi, \varphi) = 0 \\ &\Longrightarrow \begin{Bmatrix} \Lambda\bar{\partial}\partial_E\varphi = 0 \to \bar{\partial}\varphi = 0 \to \partial_E\varphi = 0, & \text{when } \varphi \text{ is a } (0,p)\text{-form} \\ \Lambda\partial_E\bar{\partial}\varphi = 0 \to \partial_E\varphi = 0, & \text{when } \varphi \text{ is a } (p,0)\text{-form}\end{Bmatrix} \Longrightarrow \partial_E\varphi = 0.\end{aligned}$$

Then by (2), $e(\Theta)\varphi = \Theta \wedge \varphi = 0$.

For p.53. Since in the strong sense $\Theta = \Theta^\rho_{\sigma i\bar{j}}\tau^\sigma_i\bar{\tau}^\rho_j \geqslant 0$ if E is positive semidefinite and let

$\varphi = \sum \varphi_A \mathrm{d}z^A$ be the local representation of the $(p,0)$-form, then we have

$$\begin{aligned}
\Theta \wedge \varphi &= \sum_{i\in A} \left(\sum \Theta^{\rho}_{\sigma i\bar{j}}\ \tau_i^\sigma \bar{\tau}_j^\rho \wedge \varphi_{A(i)} \mathrm{d}z^{A(i)} \right) = \sum_{i\in A} \left(\sum \Theta^{\rho}_{\sigma i\bar{j}} \varphi_{A(i)}\ \tau_i^\sigma \wedge \bar{\tau}_j^\rho \wedge \mathrm{d}z^{A(i)} \right) \\
&= \sum_{\rho} \left(\sum_{i\in A} \sum_{\sigma} \Theta^{\rho}_{\sigma i\bar{j}} \varphi^{\sigma}_{A(i)} (-1)^{p(i)}\ \tau_i^\sigma \wedge \mathrm{d}z^{A(i)} \wedge \bar{\tau}_j^\rho \right) \\
&= \sum_{\rho} \left[\left(\sum_{i\in A} \sum_{\sigma} \Theta^{\rho}_{\sigma i\bar{j}} \varphi^{\sigma}_{A(i)} (-1)^{p(i)} \right)\ \tau_i^\sigma \wedge \mathrm{d}z^{A(i)} \wedge \bar{\tau}_j^\rho \right] \\
&= 0,
\end{aligned}$$

which implies that

$$\sum_{i\in A} \sum_{\sigma} \Theta^{\rho}_{\sigma i\bar{j}} \varphi^{\sigma}_{A(i)} (-1)^{p(i)} = 0.$$

1.6.2 Computation of the quadratic form $\Theta(\zeta^0, \eta)$

For $L(T^*P)$ its Ricci curvature is

$$\Theta(\xi, \eta) = \begin{pmatrix} -\dfrac{\partial^2 \log h(X,X)}{\partial z^i \partial \bar{z}^j} & -\dfrac{\partial^2 \log h(X,X)}{\partial z^i \partial \bar{\xi}^\beta} \\ -\dfrac{\partial^2 \log h(X,X)}{\partial \xi^\alpha \partial \bar{z}^j} & -\dfrac{\partial^2 \log h(X,X)}{\partial \xi^\alpha \partial \bar{\xi}^\beta} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial^2 \log h_{\alpha\beta}}{\partial z^i \partial \bar{z}^j} & 0 \\ 0 & -\delta^\alpha_\beta \end{pmatrix}$$

For the Fubini-Study metric on $P_n(\boldsymbol{C})$ with holomorphic sectional curvature c, the curvature tensor is given by

$$K_{i\bar{j}k\bar{l}} = -\frac{c}{2}(h_{i\bar{j}} h_{k\bar{l}} + h_{i\bar{l}} h_{k\bar{j}}).$$

Given a point o in $P_n(\boldsymbol{C})$, we may always choose a local coordinate system around o so that the metric tensor $h_{i\bar{j}}$ coincides with δ_{ij} at o. Then the curvature of the cotangent bundle is given by

$$\frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}).$$

Note that the sign changes when we pass from TP to $T*P$. The matrix representing the Ricci curvature of $L(T^*P)$ in the proof of Proposition 6.3 reduces in this case to the following:

$$\begin{pmatrix} -\frac{c}{2}(\delta_{ij} + \delta_{in}\delta_{jn}) & 0 \\ 0 & -\delta_{\alpha\beta} \end{pmatrix}_{\substack{i,j = 1,\cdots,n \\ \alpha,\beta = 1,\cdots,n-1}}.$$

The Ricci curvature of $L(T^*P)^{-1}$ is obtained from that of $L(T^*P)$ by changing its sign. On the other hand, the Ricci curvature of P at o (which is nothing but the Ricci curvature of $K_P^{-1} = H^{n+1}$) is given by $K_{i\bar{j}} = -\frac{c}{2}(n+1)\delta_{ij}$. Hence, the Ricci curvature of H is given by

$$-\frac{1}{n+1} K_{i\bar{j}} = \frac{c}{2}\delta_{ij}.$$

Consequently, the Ricci curvature of $L(T^*P)^{-k} \otimes \pi^* H^m$ can be expressed by the following matrix:

$$\begin{pmatrix} k\frac{c}{2}(\delta_{ij} + \delta_{in}\delta_{jn}) & 0 \\ 0 & k\delta_{\alpha\beta} \end{pmatrix} + \begin{pmatrix} m\frac{c}{2}\delta_{ij} & 0 \\ 0 & 0 \end{pmatrix},$$

which is clearly positive if $m+k \geqq 1$ and $k \geqq 1$.

Now we compute the quadratic form $\Theta(\zeta^0,\eta)$ of $S^k(\mathrm{TP}) \otimes H^m$ as follow: [4, P52]

$$\begin{aligned}
\Theta(\eta,\zeta^0) &= \sum_{\alpha,\beta,i,j}\left(\begin{pmatrix} k\frac{c}{2}(\delta_{ij}+\delta_{in}\delta_{jn}) & 0 \\ 0 & k\delta_{\alpha\beta}\end{pmatrix} + \begin{pmatrix} m\frac{c}{2}\delta_{ij} & 0 \\ 0 & 0\end{pmatrix}\right)\begin{pmatrix}\eta^i\overline{\eta^j} \\ \zeta^\alpha\overline{\zeta^\beta}\end{pmatrix} \\
(\delta_{\alpha\beta}\equiv 0) &= \sum_{\alpha,\beta,i,j}\begin{pmatrix} k\frac{c}{2}(\delta_{ij}+\delta_{in}\delta_{jn}) + m\frac{c}{2}\delta_{ij} & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\eta^i\overline{\eta^j} \\ \zeta^\alpha\overline{\zeta^\beta}\end{pmatrix} \\
&= \sum_{i,j}\left[k\frac{c}{2}(\delta_{ij}+\delta_{in}\delta_{jn}) + m\frac{c}{2}\delta_{ij}\right](\eta^i\overline{\eta^j}) \\
&= k\frac{c}{2}\left(\sum_{i,j\leqslant n-1}\delta_{\mathrm{ij}}\eta^i\overline{\eta^j} + \sum_i \delta_{\mathrm{in}}\eta^i\overline{\eta^n} + \sum_j \delta_{\mathrm{jn}}\eta^n\overline{\eta^j}\right) + m\frac{c}{2}\sum_{i,j}\delta_{\mathrm{ij}}\eta^i\overline{\eta^j} \\
&= (k+m)\frac{c}{2}\sum_{i,j\leqslant n-1}\delta_{\mathrm{ij}}\eta^i\overline{\eta^j} + (2k+m)\frac{c}{2}\eta^n\overline{\eta^n} \\
&= (k+m)\frac{c}{2}\sum_{i\leqslant n-1}\eta^i\overline{\eta^i} + (2k+m)\frac{c}{2}\eta^n\overline{\eta^n}
\end{aligned}$$

which is semi-positive if $m+k \geqq 1$ and $k \geqq 1$.

Sec hyperplane Theorem

We note that the fibre of S^kE^* over $x \in M$ is the space of homogeneous polynomials of degree k on E_x.

Theorem 1.6.11. (*Theorem 1.3. of [4]*)

Let E be positive semidefinite (or negative semidefinite) of rank k at each point z of a compact Kähler manifold M. Then

$$H^q(M,\Omega^p(E)) = 0 \quad \text{if} \quad p+q \geqslant 2n-(k-r).$$

(respectively, $H^q(M,\Omega^p(E)) = 0$ if $p+q \leqslant k-r$. (By Serre Duality Theorem.))

Theorem 1.6.12. (*Lefschetz theorem on hyperplane sections*)

If the Chern class $C_{\mathbb{R}}([S]) \in H^2(M,\mathbb{R})$ of $[S]$ contains a form $\mathcal{X} \geqslant 0$ of rank k, then

$$\rho_S^* \text{ is an isomorphism for } s \leqslant k-2$$

and

$$\rho_S^* \text{ is injective for } s = k-1.$$

Proof. Let $\boldsymbol{S}^{n-1}$ be a non-singular analytic subvariety of $\boldsymbol{V}^n$, $\mathfrak{B}_s$ the restriction of the bundle $\mathfrak{B}$ on S, and the sequence

$$0 \to \Omega'^p(\mathfrak{B}) \xrightarrow{i} \Omega^p(\mathfrak{B}) \xrightarrow{r} \Omega^p(\mathfrak{B}_s) \to 0$$

be exact. And let $\eta \in \Omega'^p(\mathfrak{B})$

$$\eta = \sum_{\alpha_1 < \cdots < \alpha_p} \eta_{\alpha_1 \ldots \alpha_p} dz^{\alpha_1} \ldots dz^{\alpha_p},$$

where we assume that $z^1 = 0$ the local equation of S. Then it is clear, that for $p \geqq 1$

$$\eta' = \sum_{1 < \alpha_2 < \cdots < \alpha_p} \left(\eta_{1\alpha_2 \ldots \alpha_p}\right)_s dz^{\alpha_2} \ldots dz^{\alpha_p}$$

belongs to $\Omega^{p-1}\{(\mathfrak{B} - \{\boldsymbol{s}\})_s\}$. If we denote by $\bar{r}$ the mapping $\Omega'^p(\mathfrak{B}) \to \Omega^{p-1}\{(\mathfrak{B} - \{\boldsymbol{s}\})_s\}$ such that

$$\bar{\boldsymbol{r}}: \quad \eta \in \Omega'^p(\mathfrak{B}) \to \eta' \in \Omega^{p-1}\{(\mathfrak{B} - \{\boldsymbol{s}\})_s\},$$

then we can easily prove that the sequence

$$0 \to \Omega^p(\mathfrak{B} - \{\boldsymbol{s}\}) \to \Omega'^p(\mathfrak{B}) \to \Omega^{p-1}\{(\mathfrak{B} - \{\boldsymbol{s}\})_s\} \to 0$$

is exact. By taking the sequence of cohomology groups corresponding to (12), we obtain the exact sequence

$$\to H^q\left(\Omega^p(\mathfrak{B} - \{\boldsymbol{s}\})\right) \to H^q\left(\Omega'^p(\mathfrak{B})\right) \to H^q\left(\Omega^{p-1}\{(\mathfrak{B} - \{\boldsymbol{s}\})_s\}\right) \to .$$

Now let us assume that $\boldsymbol{S}$ is so ample that $c(\mathfrak{B} - \{\boldsymbol{s}\})$ contains an everywhere negative definite form, then we see by theorem $1''$ that

$$H^q\left(\Omega^p(\mathfrak{B} - \{\boldsymbol{s}\})\right) \simeq 0 \quad \text{for} \quad p + q \leqq n - 1,$$

$$H^q\left(\Omega^{p-1}(\mathfrak{B} - \{\boldsymbol{s}\})_s\right) \simeq 0 \quad \text{for} \quad p + q \leqq n - 1$$

Putting these in (13), we have, if $p \geqq 1$, for $p + q \leqq n - 1$

$$H^\alpha\left(\Omega'^p(\mathfrak{B})\right) \simeq 0$$

Moreover it is also the case even when $p = 0$, because

$$\Omega'^0(\mathfrak{B}) = \Omega^0(\mathfrak{B} - \{\boldsymbol{s}\}), \quad H^q\left(\Omega^0(\mathfrak{B} - \{\boldsymbol{s}\})\right) \simeq 0 \quad \text{for } q \leqq n - 1.$$

On the other hand, taking the sequence of cohomology groups of the sequence (11), we get the exact sequence

$$\to H^q\left(\Omega'^p(\mathfrak{B})\right) \to H^q\left(\Omega^p(\mathfrak{B})\right) \to H^q\left(\Omega^p\left(\mathfrak{B}_s\right)\right) \to H^{q+1}\left(\Omega'^p(\mathfrak{B})\right) \to .$$

Theorem 1.6.13.

If the divisor $\boldsymbol{S}^{n-1}$ is so ample such that $c(\mathfrak{B} - \{\boldsymbol{s}\})$ contains an everywhere negative definite form, then for $p + q \leqq n - 1$ there exists the isomorphism

$$H^q\left(\Omega^p(\mathfrak{B})\right) \to H^q\left(\Omega^p\left(\mathfrak{B}_s\right)\right)$$

and this is an isomorphism onto or into according as $p + q \leqq n - 2$ or $p + q = n - 1$.

Consider the special case, where $\boldsymbol{V}$ is a projective variety, $\boldsymbol{S}$ a generic hyperplane section of $\boldsymbol{V}$. Taking $\mathfrak{B}$ as $\{0\}$ $(\{0\}$ the trivial bundle), then clearly $\{0\} - \{\boldsymbol{S}\}$ contains an everywhere negative definite form, so the mapping

$$H^q\left(\Omega^p(0)\right) \rightarrow H^q\left(\Omega^p\left(0_s\right)\right) \text{ for } \quad p+q \leqq n-1$$

is isomorphic. But we see by the Dolbeault's theorem

$$H^q\left(\Omega^p(0)\right) \simeq H^{p,q}(\boldsymbol{V}, C), \quad H^q\left(\Omega^p\left(0_s\right)\right) \simeq H^{p,q}(\boldsymbol{S}, C),$$

where C is complex number field. Hence we have the Lefschetz theorem in the classical form:

Theorem 1.6.14.

Let $\boldsymbol{V}$ be an algebraic variety of dim. n without singularities immersed in a projective space, $\boldsymbol{S}$ be a generic hyperplane section of it (consequently $\boldsymbol{S}$ is irreducible and has no singularities), $H(\boldsymbol{V}, C)$ the cohomology group of degree r. Then $H^r(\boldsymbol{V}, C)$ is isomorphic to $H^r(\boldsymbol{S}, C)$ if $r \leqq n-2$, and $H^{n-1}(\boldsymbol{V}, C)$ is isomorphic to a submodule of $H^{n-1}(\boldsymbol{S}, C)$.

Sec 1.7 Summary and Reflection

Vanishing theorems are furnish criteria for the non-existence of nontrivial harmonic tensor fields, to show that certain cohomology groups $H^q(X, 0)$ are zero. According to S. Bochner's general criterion, a tensor of a specified type cannot satisfy a given "harmonic " equation globally on a compact manifold, unless it is identically zero.

2 Logarithmic vanishing theorems on compact Kähler manifolds I

Part I

Sec 2.1 Terminologies

Table 2.1: Terminologies Interpretation

Terminologies	Interpretations
SNC divisor D	every irreducible component D_i is smooth and all intersections are *transverse*.
$\Omega_X^p(\log D)$	The sheaf of germs of differential p-forms on X with at most logarithmic poles along D, (introduced by Deligne in) whose sections on an open subset V of X are $\Gamma(V, \Omega_X^p(\log D)) := \{\alpha \in \Gamma(V, \Omega_X^p \otimes \mathcal{O}_X(D))$ $\&\ \ \mathrm{d}\alpha \in \Gamma(V, \Omega_X^{p+1} \otimes \mathcal{O}_X(D))\}$.
E	Holomorphic vector bundle of rank k over a complex manifold M.
h	A smooth Hermitian metric on E.
∇	The Chern connection of (E, h), which is compatible with h and complex structure on E.
$Y = X \backslash D$	The complement of a SNC divisor D in a compact Kähler manifold X.
Poincaré type metric ω_P	A smooth Kähler metric on $Y = X\backslash D$ which is of Poincaré type along D. According to [11, proposition 3.2 and 3.4], this metric is *complete* and of *finite volume*. Moreover, its *curvature tensor and covariant derivatives are bounded*. $\omega_P = \sqrt{-1}\sum_{j=1}^{k} \frac{\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j}{\lvert z_j\rvert^2 \cdot \log^2 \lvert z_j\rvert^2} + \sqrt{-1}\sum_{j=k+1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j.$
$\Omega_{(2)}^{p,q}(X, E, \omega_Y, h_Y^E)$ (or $\Omega_{(2)}^{p,q}(X, E)$)	Whose section space $\Gamma(U, \Omega_{(2)}^{p,q}(X,E))$ over $\forall U \overset{\text{open}}{\subset} X$ consists of E-valued (p,q)-forms u with measurable coefficients such that the L^2 norms of both u and $\overline{\partial} u$ are integrable on any compact subset $V \subset U$. (Local integrable)
$L_{p,q}^2(X, E)$	$L^2(X, \Lambda^{p,q}T^*M \otimes E) = L_{p,q}^2(X, E) = \mathcal{A}_{p,q}^2(X) \otimes E$.
Fine sheaf	For any finite open covering $\mathfrak{U} = \{U_j\}$, there is a family of homomorphisms $\{h_j\}$, $h_j \colon \mathscr{S} \to \mathscr{S}$, such that the support of h_j satisfying that $\mathrm{Supp}(h_j) \subset U_j$ and $\sum_j h_j =$ identity. (Partition of unity)
$\mathbb{R}$-divisor	T is called an $\mathbb{R}$-divisor if it is an element of $\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{Div}(X)$ is the set of divisors in X.
Terminologies	*Interpretations*

Continued on next page

Table 2.1: Terminologies Interpretation (Continued)

Terminologies	Interpretations
$\mathbb{R}$-linear equivalence $T_1 \sim_{\mathbb{R}} T_2$	$T_1 - T_2$ can be written as a finite sum of principal divisors with real coefficients, i.e. $T_1 - T_2 = \sum_{i=1}^{k} r_i(f_i)$, where $r_i \in \mathbb{R}$ and (f_i) is the principal divisor associated to meromorphic function f_i.
$\mathbb{R}$-line bundle L	$L = \sum_i a_i L_i$ is a finite sum with real numbers $a_1, \cdots, a_k$ and certain line bundles $L_1, \cdots, L_k$. It is *k-positive*if *there exists smooth metrics* $h_1, \cdots, h_k$ on $L_1, \cdots, L_k$ such that the curvature of the induced metric on L : $\sqrt{-1}\Theta(L, h) = \sum_{i=1}^{k} a_i \sqrt{-1}\Theta(L_i, h_i)$ is k-positive.
Terminologies	*Interpretations*

Attention!: For any *fine sheaf* $\mathscr{S}$, one has $H^q(X, \mathscr{S}) = 0$ for $q \geqslant 1$.

Sec 2.2 Some problems and their solutions

Problem 2.2.1. ***(Fine sheaf)***

Why one can obtain the fact that $\Omega_{(2)}^{p,q}(X, E)$ admits a partition of unity from the consequence that if $u \in \Gamma(U, \Omega_{(2)}^{p,q}(X, E))$ and $f \in C^\infty(X)$, then $fu \in \Gamma(U, \Omega_{(2)}^{p,q}(X, E))$?

Problem 2.2.2. ***(Splitting of holomorphic vector bundle)***

Why the metric $\widetilde{\omega}_P$ on the holomorphic tangent bundle TY is of the splitting form : $\widetilde{\omega}_P = \sum_{i=1}^{n} \omega_i(z_i)$? Then why in local computations, we can treat $(TY, \widetilde{\omega}_P)$ as a direct sum of line bundles $\bigoplus_{i=1}^{n}(F_i, \omega_i)$, i.e. $(TY, \widetilde{\omega}_P) := \bigoplus_{i=1}^{n}(F_i, \omega_i)$?

Theorem 2.2.1.

There exists a smooth Hermitian metric h_Y^L on $L|_Y$ such that the sheaf $\Omega_X^p(\log D) \otimes \mathcal{O}(L)$ enjoys a fine resolution given by the L^2-Dolbeault complex $\left(\Omega_{(2)}^{p,*}(X, L, \omega_P, h_Y^L), \bar{\partial}\right)$.

Sec 2.3 Preliminaries

2.3.1 Proof of Theorem1.1

Lemma 2.3.1. *(3.3)*

$$\left\langle \left[\sqrt{-1}\Theta(V, h^V), \Lambda_{\tilde{\omega}_P}\right] u, u \right\rangle \geqslant C|u|^2.$$

Proof.

$$\Omega_Y^p \otimes K_Y^{-1} = \Omega_Y^p \otimes \Omega_Y^{-n} = \Omega_Y^{-(n-p)} =^a \bigoplus_{i_1,\cdots,i_{n-p}} (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1}).$$

$$\begin{aligned}\Omega_Y^p \otimes \Omega_Y^{-n} &= \Omega_Y^p \otimes (\Omega_Y^{-p} \otimes \Omega_Y^{-(n-p)}) \\ &= (\Omega_Y^p \otimes 1) \otimes (\Omega_Y^{-p} \otimes \Omega_Y^{-(n-p)}) \\ &= (\Omega_Y^p \otimes 1) \otimes (\Omega_Y^{-p} \otimes 1) \otimes (1 \otimes \Omega_Y^{-(n-p)}) \\ &= 1 \otimes \Omega_Y^{-(n-p)} = \Omega_Y^{-(n-p)}.\end{aligned}$$

$$(\Omega_Y^p \otimes 1) \otimes (\Omega_Y^{-p} \otimes 1) = \Omega_Y^p(\Omega_Y^{-p}) \otimes 1 = 1(\text{trivial holomorphic cotangent bundle}) \otimes 1.$$

[a] For $(\mathrm{T}Y, \tilde{\omega}_P) = \bigoplus_{i=1}^n (F_i, \omega_i)$ by using (2.1), then in local case (Fixed a local coordinate $(W; z_1, \cdots, z_n)$) one has $F_i^* = F_i^{-1}$(Dual line bundle)[7, §2.2, p71]. (Ω_Y is the dual of TY) The conclusion is clear through some easy computation.

We have

$$\sqrt{-1}\Theta(V, h^V) = \sum_{i_1,\cdots,i_{n-p}} \sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})]$$

and

$$\begin{aligned}\sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})] &= \sum_i \sqrt{-1}\partial\overline{\partial}\log(\omega_i) \\ (3.12) &\geqslant \alpha\tilde{\omega}_P - (n-p)C\tilde{\omega}_P \\ (\text{if } \alpha > (n-p+1)C) &> C\tilde{\omega}_P > 0,\end{aligned}$$

where we denote

$$\sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})] = \sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1}), h_Y^L \otimes \omega_{i_1}^* \otimes \cdots \otimes \omega_{i_{n-p}}^*].$$

Thus, the curvature of each summand of $L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})$ is strictly positive, i.e.

$$\sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})] > 0.$$

$$\begin{aligned}\left\langle \left[\sqrt{-1}\Theta(V, h^V), \Lambda_{\tilde{\omega}_P}\right] u, u \right\rangle &= \left\langle \left[\sum_{i_1,\cdots,i_{n-p}} \sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})], \Lambda_{\tilde{\omega}_P} \right] u, u \right\rangle \\ &\geqslant (q \cdot C - (n-n))|u|^2 \geqslant C|u|^2. (q \geqslant 1)\end{aligned}$$

And the proof of the assertion that of the metric $\widetilde{h}_Y^L$ is Nakano positive is on following.

$$\begin{aligned}
&\sum_{i_1,\cdots,i_{n-p}} \sqrt{-1}\Theta[L|_Y \otimes (F_{i_1}^{-1} \otimes \cdots \otimes F_{i_{n-p}}^{-1})] \\
&= \sum_{i,j,\alpha,\gamma} \sqrt{-1}R_{i\bar{j}\alpha}^{\gamma}\, \mathrm{dz}^i \wedge d\bar{z}^j \otimes e^{\alpha} \otimes e_{\gamma} (\text{cf P.5}) \\
(R_{i\bar{j}\alpha}^{\gamma} = h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}) &= \sum_{i,j,\alpha,\beta} \sqrt{-1}h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}}\, \mathrm{dz}^i \wedge d\bar{z}^j \otimes e^{\alpha} \otimes \bar{e}^{\beta} \\
\mathrm{dz}^i = \left(u^i \frac{\partial}{\partial z^i}\right) &= \sum_{i,j,\alpha,\beta} \sqrt{-1}h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}} \left(u^i \frac{\partial}{\partial z^i}\right) \wedge \left(\bar{u}^j \frac{\partial}{\partial \bar{z}^j}\right) \otimes e^{\alpha} \otimes \bar{e}^{\beta} \\
&= \sum_{i,j,\alpha,\beta} \sqrt{-1}h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} \left(\frac{\partial}{\partial z^i} \otimes e^{\alpha}\right) \wedge \left(\frac{\partial}{\partial \bar{z}^j} \otimes \bar{e}^{\beta}\right) \\
&> \sum_{i_1,\cdots,i_{n-p}} C\widetilde{\omega}_P > 0
\end{aligned}$$

As $h^{\gamma\bar{\beta}} > 0$, then we have $\sum_{i,j,\alpha,\beta} \sqrt{-1}h^{\gamma\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha}\bar{u}^{j\beta} > 0$, which immediately shows that

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0.$$

Thus $\widetilde{h}_Y^L$ is Nakano positive. ■

Theorem 2.3.2. (*Main theorem*)

Let

X	*A compact Kähler manifold*
D	*A small normal crossing (SNC) divisor*
N	*A line bundle*
$\Delta = \sum_{i=1}^{s} \alpha_i D_i$	*an $\mathbb{R}$-divisor with $\alpha_i \in [0,1]$ such that $N \otimes \mathcal{O}_X([\Delta])$ is a k-positive $\mathbb{R}$-line bundle*
L	*A nef line bundle*

Then we have

$$H^q(X, \Omega_X^p(\log D) \otimes L \otimes N) = 0, \text{ for any } p + q \geqslant n + k + 1.$$

***Proof* (A small sketch).** The following computation is of (4.5).

$$
\begin{aligned}
&\sqrt{-1}\Theta(\mathcal{F}, h^{\mathcal{F}}_{\alpha,\varepsilon,\tau})\\
&= \sqrt{-1}\Theta(L \otimes F \otimes \mathcal{O}_X(-[\Delta]), h^{\mathcal{F}}_{\alpha,\varepsilon,\tau})\\
&= \sqrt{-1}\partial\overline{\partial}\log\big(h^{\mathcal{F}}_{\alpha,\varepsilon,\tau}\big)\\
&= \sqrt{-1}\partial\overline{\partial}\log\big(h^{L}\big) + \sqrt{-1}\partial\overline{\partial}\log\big(h^{F}\big) + \sqrt{-1}\partial\overline{\partial}\log\big(h^{\Delta}\big)^{-1} + \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}\|\sigma_i\|^{2\tau_i}_{D_i}(\log^2(\varepsilon\|\sigma_i\|^2_{D_i}))^{\frac{\alpha}{2}}\right)\\
&= \sqrt{-1}\partial\overline{\partial}\log\big(h^{L}\big) + \sqrt{-1}\partial\overline{\partial}\log\big(h^{F}\big) + \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}h^{a_i}_{D_i}\right)^{-1} + \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}\|\sigma_i\|^{2\tau_i}_{D_i}\right)\\
&+ \sqrt{-1}\partial\overline{\partial}\log\left(\prod_{i=1}^{s}(\log^2(\varepsilon\|\sigma_i\|^2_{D_i}))^{\frac{\alpha}{2}}\right)\\
&= \sqrt{-1}\partial\overline{\partial}\log\big(h^{L}\big) + \sqrt{-1}\partial\overline{\partial}\log\big(h^{F}\big) - \sum_{i=1}^{s}a_i\sqrt{-1}\partial\overline{\partial}\log(h_{D_i}) + \sum_{i=1}^{s}\tau_i\sqrt{-1}\partial\overline{\partial}\log\big(\|\sigma_i\|^2_{D_i}\big)\\
&+ \sum_{i=1}^{s}\alpha\sqrt{-1}\partial\overline{\partial}\log\big(\log(\varepsilon\|\sigma_i\|^2_{D_i})\big)\\
&= \sqrt{-1}\partial\overline{\partial}\log\big(h^{L}\big) + \sqrt{-1}\partial\overline{\partial}\log\big(h^{F}\big) - \sum_{i=1}^{s}a_i c_1(D_i) + \sum_{i=1}^{s}\tau_i\sqrt{-1}\partial\overline{\partial}\log(h_{D_i})\\
&+ \sum_{i=1}^{s}\alpha\sqrt{-1}\partial\left(\frac{\overline{\partial}\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}{\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}\right)\\
&= \sqrt{-1}\partial\overline{\partial}\log\big(h^{L}\big) + \sqrt{-1}\partial\overline{\partial}\log\big(h^{F}\big) - \sum_{i=1}^{s}a_i c_1(D_i) + \sum_{i=1}^{s}\tau_i c_1(D_i)\\
&+ \sum_{i=1}^{s}\alpha\sqrt{-1}\left(\frac{\partial\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)\cdot\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right) - \overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)\wedge\partial\log\left(\|\sigma_i\|^2_{D_i}\right)}{\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right)^2}\right)\\
&= \sqrt{-1}\Theta(L, h^L) + \sqrt{-1}\Theta(F, h^F) + \sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) + \sum_{i=1}^{s}\left(\frac{\alpha\sqrt{-1}\partial\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)}{\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}\right)\\
&+ \sqrt{-1}\sum_{i=1}^{s}\left(\frac{\alpha\partial\log\left(\|\sigma_i\|^2_{D_i}\right)\wedge\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)}{\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right)^2}\right)\\
&= \sqrt{-1}\Theta(L, h^L) + \sqrt{-1}\Theta(F, h^F) + \sum_{i=1}^{s}(\tau_i - a_i)c_1(D_i) + \sum_{i=1}^{s}\left(\frac{\alpha c_1(D_i)}{\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)}\right)\\
&+ \sqrt{-1}\sum_{i=1}^{s}\left(\frac{\alpha\partial\log\left(\|\sigma_i\|^2_{D_i}\right)\wedge\overline{\partial}\log\left(\|\sigma_i\|^2_{D_i}\right)}{\left(\log\left(\varepsilon\|\sigma_i\|^2_{D_i}\right)\right)^2}\right).
\end{aligned}
$$

where $h_{D_i} = \|\sigma_i\|^2_{D_i}$.

Remark.

$$\begin{aligned}\mathcal{O}_X([D]) \otimes \mathcal{O}_X(-[\Delta]) &= \mathcal{O}_X(\sum_{i=1}^{s}[D_i]) \otimes \mathcal{O}_X(-\sum_{i=1}^{s} a_i[D_i]) \\ (\textit{Dual line bundle}) &= \sum_{i=1}^{s} a_i \mathcal{O}_X([D_i]) \otimes \mathcal{O}_X(-[D_i]) \\ &= \sum_{i=1}^{s} a_i \mathcal{O}_X(1).\ (\textit{trivial line bundle})\end{aligned}$$

Definition 2.3.1. (*Poincaré Type Metric*)

A metric ω_Y is of Poincaré Type along D if for each local coordinate chart $(W; z_1, \cdots, z_n)$ along D, the restriction $\omega|_{W^*_{1/2}}$ is equivalent to the usual Poincaré type metric ω_P defined by

$$\omega_P = \sqrt{-1}\sum_{i=1}^{k} \frac{\mathrm{d}z_j \wedge d\overline{z_j}}{|z_j|^2 \cdot \log^2 |z_j|^2} + \sqrt{-1}\sum_{j=k+1}^{n} \mathrm{d}z_j \wedge d\overline{z_j}.$$

Where $W^*_r = Y \cap W = (\Delta^*_r)^k \times (\Delta_r)^{n-k}, r \in (0, \frac{1}{2}]$.

Theorem 2.3.3. (*The key theorem for the proof of the main theorem*)

Let

(X, ω)	*A compact Kähler manifold of dimension* n
D	*A SNC divisor in X*
ω_P	*A smooth Kähler metric on* $Y = X - D$ *which is Poincaré Type along* D

Then there exists a smooth Hermitian metric h^L_Y *on* $L|_Y$ *such that the sheaf* $\Omega^p(\log D) \otimes \mathcal{O}(L)$ *over* X *has a fine resolution given by the* L^2 *Dolbeault complex* $(\Omega^{p,*}_{(2)}(X, L, \omega_P, h^L_Y), \overline{\partial})$.
In other words, we have an ***exact sequence of sheaf over*** X

$$0 \to \Omega^p(\log D) \otimes \mathcal{O}(L) \to \Omega^{p,*}_{(2)}(X, L, \omega_P, h^L_Y)$$

such that $\Omega^{p,q}_{(2)}(X, L, \omega_P, h^L_Y)$ *is a* ***fine sheaf*** *for any* $0 \leq p, q \leq n$*. In particular,*

$$H^q(X, \Omega^p(\log D) \otimes \mathcal{O}(L)) \cong H^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y) \cong \mathbb{H}^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y). \tag{2.1}$$

Note: *The isomorphism holds up to equivalence of metrics, i.e. if* $\widetilde{\omega}_P \sim \omega_P$ *and* $\widetilde{h}^L_Y \sim h^L_Y$*, then*

$$\mathbb{H}^{p,q}_{(2)}(Y, L, \omega_P, h^L_Y) \cong \mathbb{H}^{p,q}_{(2)}(Y, L, \widetilde{\omega}_P, \widetilde{h}^L_Y).$$

! Replacing the line bundle with vector bundle is still valid.

Problem 2.3.3. ***(Why $\Omega^{p,q}_{(2)}(X,E)$ is a fine sheaf over X?)***

In the paper [6, §2.3, P7], the author asserts that if $u \in \Gamma(U, \Omega^{p,q}_{(2)}(X,E))$ and $f \in C^\infty(X)$, then $fu \in \Gamma(U, \Omega^{p,q}_{(2)}(X,E))$. This demonstrates the existence of a partition of unity in $\Omega^{p,q}_{(2)}(X,E)$. Subsequently, the author claims that $\Omega^{p,q}_{(2)}(X,E)$ is a fine sheaf over X.

Proof. The basis for this assertion lies in the properties of fine sheaves and the specific construction of $\Omega^{p,q}_{(2)}(X,E)$:

1. **Definition of a Fine Sheaf**: *A sheaf $\mathcal{F}$ is considered "fine" if it satisfies certain partition of unity properties.* In this context, it means that for any open cover $\{U_i\}$ of the underlying topological space X, there exist smooth functions $\rho_i \in C^\infty(X)$ with specific properties:
   - $0 \leq \rho_i \leq 1$ for all i.
   - $\mathrm{supp}(\rho_i) \subseteq U_i$ (the support of ρ_i is contained in U_i).
   - $\sum \rho_i(x) = 1$ for all x in X (the sum of ρ_i at each point x is 1).

   These partition of unity functions ρ_i are crucial for gluing together local sections of the sheaf to obtain global sections.
2. **Construction of $\Omega^{p,q}_{(2)}(X,E)$**: This sheaf represents smooth differential forms of type (p,q) with values in a vector bundle E over the manifold X. *Its construction involves defining local sections on coordinate patches and specifying how these sections transition between overlapping patches.*
3. **Demonstrating Fine Sheaf Property**: In Section 2.3 of the paper, the author asserts that if u is a section in $\Omega^{p,q}_{(2)}(X,E)$ and f is a smooth function on X, then the product fu is also a section in $\Omega^{p,q}_{(2)}(X,E)$. *This demonstrates compatibility with the fine sheaf property because it shows that you can use smooth functions (such as the ρ_i functions from the partition of unity) to combine sections locally without leaving the sheaf $\Omega^{p,q}_{(2)}(X,E)$.*

   Essentially, this step ensures that $\Omega^{p,q}_{(2)}(X,E)$ is closed under multiplication by smooth functions, which is a crucial property for a sheaf to be fine.

■

Remark. *In summary,* the assertion that $\Omega^{p,q}_{(2)}(X,E)$ is a fine sheaf is based on the construction of $\Omega^{p,q}_{(2)}(X,E)$ and the demonstration that it satisfies the necessary partition of unity property when sections are multiplied by smooth functions. *This property is essential for many purposes in differential geometry and* allows for the gluing of local sections to obtain global sections over a manifold X.

Problem 2.3.4.

1. *How to get (3.5)?*
2. *How to obtain the Laurentz series representation of $\sigma_I(z)$ on $W^*_{1/2}$?*
3. *Why σ is L^2 integrable on W^*_r iff $\beta_j > -\tau_j$ along D_j by using polar coordinates and Fubini Theorem (Example 2.4)?*
4. *Why σ and $\nabla\sigma$ have only logarithimic pole and σ is a section of $\Omega^p(\log D) \otimes \mathcal{O}(L)$ on W?*

Solution. 1.

2. The Laurentz series equation for several variables is

$$f(z_1, \cdots, z_n) = \sum_{J=-\infty}^{\infty} a_J (z_1 - z_{1_0})^{j_1} \cdots (z_n - z_{n_0})^{j_n}, \quad R_1 \leqslant |z_i - z_{i_0}| \leqslant R_2,$$

where $J=(j_1,\cdots,j_n)$. $f(z_1,\cdots,z_n)$ is single-valued analysis in the annulus centered at every point z_{i_0}. The coefficients are

$$a_J=\frac{1}{(2\pi i)^n}\int_{\Omega_1}\cdots\int_{\Omega_n}\frac{f(z_1,\cdots,z_n)}{(z_1-z_{1_0})^{j_1+1}\cdots(z_n-z_{n_0})^{j_n+1}}\mathrm{d}z_J,$$

where $\mathrm{d}z_J=\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$ and $\Omega_1,\cdots,\Omega_n$ are counter-clockwise closed curves surrounding the expansion point $(z_{1_0},\cdots,z_{n_0})$ in each variable, and the order of inegration can be interchanged. By using the above equation, for a fixed point $(0,\cdots,0)\in W^*_{1/2}=\Delta^{*t}_{1/2}\times\Delta^{n-t}_{1/2}$, we have

$$\sigma_I(z)=\sum_{J=-\infty}^{\infty}a_J(z_1)^{j_1}\cdots(z_t)^{j_t},\quad J=(j_1,\cdots,j_t),$$

where $a_J=\sigma_{IJ}(z_{t+1},\cdots,z_n)$ is a holomorphic function on $\Delta^{n-t}_{1/2}$. Thus $\sigma_I(z)$ is bounded on $W^*_r\subset W^*_{1/2}$, i.e. there exists a positive constant M such that $|\sigma_I(z)|\leqslant M$.

3. By using polar coordinates, we obtain that

$$\begin{aligned}
&\|\sigma\|^2_{L^2(W^*_r)}\\
&=\sum_{|I|=p}\int_{W^*_r}|e|^2_{h^L}\left(|\sigma_I(z)|^2\prod_{\nu=1}^{b}\log^2|z_{i_{p\nu}}|^2\prod_{i=1}^{t}|z_i|^{2\tau_i}(\log^2|z_i|^2)^{\alpha/2}\right)\omega_P^n\\
&\leqslant\sum_{|I|=p}\int_{W^*_r}\left(|\sigma_I(z)|^2\prod_{\nu=1}^{b}\log^2|z_{i_{p\nu}}|^2\prod_{i=1}^{t}|z_i|^{2\tau_i}(\log^2|z_i|^2)^{\alpha/2}\right)\omega_P^n\\
&\leqslant\sum_{|I|=p}\Bigg(\underbrace{\int_0^{2\pi}\cdots\int_0^{2\pi}}_{b+t}\Bigg)\Bigg(\underbrace{\int_0^{\frac12}\cdots\int_0^{\frac12}}_{b+t}\Bigg)\left(|\sigma_I(z)|^2\prod_{\nu=1}^{b}\log^2 r^2_{i_{p\nu}}\prod_{i=1}^{t}r_i^{2\tau_i}(\log^2 r_i^2)^{\alpha/2}\right)\omega_P^n\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{r}\\
&=\sum_{|I|=p}|\sigma_I(z)|^2\prod_{\nu=1}^{b}\left(\int_0^{2\pi}\int_0^{\frac12}\log^2 r^2_{i_{p\nu}}\mathrm{d}\theta_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\prod_{i=1}^{t}\left(\int_0^{2\pi}\int_0^{\frac12}r_i^{2\tau_i}(\log^2 r_i^2)^{\alpha/2}\mathrm{d}\theta_i\mathrm{d}r_i\right)\omega_P^n\\
&=\sum_{|I|=p}(2\pi)^{b+t}|\sigma_I(z)|^2\prod_{\nu=1}^{b}\left(\int_0^{\frac12}\log^2 r^2_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{\frac12}r_i^{2\tau_i}(\log^2 r_i^2)^{\alpha/2}\mathrm{d}r_i\right)\omega_P^n\\
&=\sum_{|I|=p}(2\pi)^{b+t}2^{2b+\alpha t}|\sigma_I(z)|^2\prod_{\nu=1}^{b}\left(\int_0^{\frac12}\log^2 r_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{\frac12}r_i^{2\tau_i}(\log r_i)^{\alpha}\mathrm{d}r_i\right)\omega_P^n\\
&\leqslant\sum_{|I|=p}(2\pi)^{b+t}2^{2b+\alpha t}M^2\left[\prod_{\nu=1}^{b}\left(\int_0^{\frac12}\log^2 r_{i_{p\nu}}\mathrm{d}r_{i_{p\nu}}\right)\cdot\prod_{i=1}^{t}\left(\int_0^{\frac12}r_i^{2\tau_i}(\log r_i)^{\alpha}\mathrm{d}r_i\right)\right]\left(\frac12\right)^n\\
&<+\infty\quad\text{(By using Example 2.4)}
\end{aligned}$$

where $|e|^2_{h^L}\in[\frac12,1]$ over W by hypothesis and $r_i=|z_i|$, $\mathrm{d}\boldsymbol{r}=\mathrm{d}r_{i_{p1}}\wedge\cdots\wedge\mathrm{d}r_{i_{pb}}\wedge\mathrm{d}r_1\wedge\cdots\wedge\mathrm{d}r_t$, $\mathrm{d}\boldsymbol{\theta}=\mathrm{d}\theta_{i_{p1}}\wedge\cdots\wedge\mathrm{d}\theta_{i_{pb}}\wedge\mathrm{d}\theta_1\wedge\cdots\wedge\mathrm{d}\theta_t$. $(r\le\frac12)$ Thus σ is L^2 integrable on W^*_r iff $\beta_j>-\tau_j$ along D_j.

4. We have

$$
\begin{aligned}
\nabla\sigma(z) &= \sum_{|I|=p} \nabla\left(\sigma_I(z)\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes e\right) \\
&= \sum_{|I|=p} \mathrm{d}\sigma_I(z)\wedge\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes e + \sum_{|I|=p}\sigma_I(z)\wedge\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes \mathrm{d}e \\
&+ \sum_{\nu=1}^{p}\left(\sum_{|I|=p}\sigma_I(z)\zeta_{i_1}\wedge\cdots\wedge(\mathrm{d}\zeta_{i_\nu})\wedge\cdots\wedge\zeta_{i_p}\otimes e\right) \\
&= \sum_{|I|=p} \mathrm{d}\sigma_I(z)\wedge\zeta_{i_1}\wedge\cdots\wedge\zeta_{i_p}\otimes e
\end{aligned}
$$

,where

$$
\begin{aligned}
\mathrm{d}\sigma_I(z) &= \sum_{J=-\infty}^{\infty}\mathrm{d}\left(a_J(z_1)^{j_1}\cdots(z_t)^{j_t}\right) \\
&= \sum_{J=-\infty}^{\infty}\mathrm{d}\left(a_J\right)(z_1)^{j_1}\cdots(z_t)^{j_t} + \sum_{J=-\infty}^{\infty} a_J\mathrm{d}\left((z_1)^{j_1}\cdots(z_t)^{j_t}\right) \\
&= \sum_{J=-\infty}^{\infty}\mathrm{d}\left(\sigma_{IJ}(z_{t+1},\cdots,z_n)\right)(z_1)^{j_1}\cdots(z_t)^{j_t} \\
&+ \sum_{J=-\infty}^{\infty}\sigma_{IJ}(z_{t+1},\cdots,z_n)\mathrm{d}\left((z_1)^{j_1}\cdots(z_t)^{j_t}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sigma_{IJ}(z_{t+1},\cdots,z_n) \\
&= \frac{1}{(2\pi i)^{n-t}}\int_{W^*_{1/2}}\frac{\sigma_I(z_{t+1},\cdots,z_n)}{(z_{t+1}-z_{t+1_0})^{j_{t+1}+1}\cdots(z_n-z_{n_0})^{j_n+1}}\mathrm{d}z_{t+1}\wedge\cdots\wedge\mathrm{d}z_n.
\end{aligned}
$$

As $\sigma_{IJ}(z_{t+1},\cdots,z_n)$ is a holomorphic function on $\Delta^{n-t}_{1/2}$, thus it has only removable singularity, and so as to $\sigma_I(z)$, which shows that σ and $\nabla\sigma$ have only logarihmic pole.

■

Sec Additional Material

Definition of Nef line bundle

Definition 2.3.2. ***(Nef line bundle (Algebraic version))***

More generally, a line bundle L on a proper scheme X over a field k is said to be nef if it has nonnegative degree on every (closed irreducible) curve in X (The degree of a line bundle L on a proper curve C over k is the degree of the divisor (s) of any nonzero rational section s of L.) A line bundle may also be called an invertible sheaf.

The term "nef" was introduced by Miles Reid as a replacement for the older terms "arithmetically effective" (Zariski 1962) and "numerically effective", as well as for the phrase "numerically eventually free". The older terms were misleading, in view of the examples below.

Every line bundle L on a proper curve C over k which has a global section that is not identically zero has nonnegative degree. As a result, a basepoint-free line bundle on a proper scheme X over k has nonnegative degree on every curve in X; that is, it is nef. More generally, a line bundle L is called semi-ample if some positive tensor power $L^{\otimes a}$ is basepoint-free. It follows that a semi-ample line bundle is nef. Semi-ample line bundles can be considered the main geometric source of nef line bundles, although the two concepts are not equivalent; see the examples below.

A Cartier divisor D on a proper scheme X over a field is said to be nef if the associated line bundle $O(D)$ is nef on X. Equivalently, D is nef if the intersection number $D \cdot C$ is nonnegative for every curve C in X.

To go back from line bundles to divisors, the first Chern class is the isomorphism from the Picard group of line bundles on a variety X to the group of Cartier divisors modulo linear equivalence. Explicitly, the first Chern class $\mathrm{C}_1(\mathrm{L})$ is the divisor (s) of any nonzero rational section s of L.

The nef cone

To work with inequalities, it is convenient to consider $\mathbf{R}$-divisors, meaning finite linear combinations of Cartier divisors with real coefficients. The $\mathbf{R}$-divisors modulo numerical equivalence form a real vector space $N^1(X)$ of finite dimension, the Néron-Severi group tensored with the real numbers. (Explicitly: two $\mathbf{R}$-divisors are said to be numerically equivalent if they have the same intersection number with all curves in X.) An $\mathbf{R}$-divisor is called nef if it has nonnegative degree on every curve. The nef $\mathbf{R}$-divisors form a closed convex cone in $N^1(X)$, the nef cone $\mathrm{Nef}(X)$.

The cone of curves is defined to be the convex cone of linear combinations of curves with nonnegative real coefficients in the real vector space $N_1(X)$ of 1-cycles modulo numerical equivalence. The vector spaces $N^1(X)$ and $N_1(X)$ are dual to each other by the intersection pairing, and the nef cone is (by definition) the dual cone of the cone of curves.

A significant problem in algebraic geometry is to analyze which line bundles are ample, since that amounts to describing the different ways a variety can be embedded into projective space. One answer is Kleiman's criterion (1966): for a projective scheme X over a field, a line bundle (or $\mathbf{R}$-divisor) is ample if and only if its class in $N^1(X)$ lies in the interior of the nef cone. (An $\mathbf{R}$-divisor is called ample if it can be written as a positive linear combination of ample Cartier divisors.) It follows from Kleiman's criterion that, for X projective, every nef $\mathbf{R}$-divisor on X is a limit of ample $\mathbf{R}$-divisors in $\mathrm{N}^1(\mathrm{X})$. Indeed, for D nef and A ample, $D + cA$ is ample for all real numbers $c > 0$.

Definition 2.3.3. (***Metric definition of nef line bundles (Geometry version)***)

Let X be a compact complex manifold with a fixed Hermitian metric, viewed as a positive $(1,1)$-form ω. Following Jean-Pierre Demailly, Thomas Peternell and Michael Schneider, **a holomorphic line bundle L on X is said to be nef if for every $\varepsilon > 0$ there is a smooth Hermitian metric h_ε on L whose curvature satisfies $\Theta_h(L) \geqslant -\varepsilon\omega$.** When X is projective over C, this is equivalent to the previous definition (that L has nonnegative degree on all curves in X) which explains the more complicated definition just given.

Definition 2.3.4. ***(Logarithmic pole)***

For a complex function $f(z)$, if there exists a pole at z_0 with the following form:

$$f(z) \sim \frac{C}{(z - z_0)\log(z - z_0)},$$

where $\sim$ denotes that the ratio tends to 1 as $z \to z_0$, C is a nonzero complex number, and $\log(z - z_0)$ represents the logarithmic function, then z_0 is called a logarithmic pole of the function $f(z)$.

Note that the characteristic of a logarithmic pole is that the function becomes very large in magnitude as we approach points near z_0.

L^2-APPROACH TO THE SAITO VANISHING THEOREM

3

Part I

Sec 3.1 Terminologies

Table 3.1: Terminologies Interpretation

Terminologies	Interpretations
A variation of Hodge structures on X with weight k	A smooth vector bundle E on X with decomposition $E = \bigoplus_{p+q=k} E^{p,q}$ by smooth vector bundles and a flat connection $\nabla : E \to \mathcal{A}^1_X(E)$ that maps each $\mathcal{A}_X(E^{p,q})$ into $\mathcal{A}^{1,0}_X(E^{p,q}) \otimes \mathcal{A}^{1,0}_X(E^{p-1,q+1}) \otimes \mathcal{A}^{0,1}_X(E^{p,q}) \otimes \mathcal{A}^{0,1}_X(E^{p+1,q-1})$. $(\nabla^2 = 0)$
Filtration on E	$F^pE = \bigoplus_{p' \geqslant p} E^{p',k-p'}$ and $\nabla^{1,0} : \mathcal{A}^{r,s}_X(E^{p,q}) \to \mathcal{A}^{r+1,s}_X(E^{p,q})$; $\theta : \mathcal{A}^{r,s}_X(E^{p,q}) \to \mathcal{A}^{r+1,s}_X(E^{p-1,q+1})$ $\overline{\partial} : \mathcal{A}^{r,s}_X(E^{p,q}) \to \mathcal{A}^{r,s+1}_X(E^{p,q})$; $\varphi : \mathcal{A}^{r,s}_X(E^{p,q}) \to \mathcal{A}^{r+1,s}_X(E^{p+1,q-1})$
A polarization on E	A sesquilinear pairing $Q : E \otimes \overline{E} \to \mathcal{A}_X$ such that 1. Q is compatible with ∇ in the sense that $\nabla Q(u,v) = Q(\nabla u, v) + Q(u, \nabla v)$ for all smooth sections u, v of E, 2. The summands $E^{p,q}$ od the decomposition are mutually orthogonal to each other, 3. $h(v,w) = \sum_{p+q=k}(-1)^q Q(v^{p,q}, w^{p,q})$ is positive definite.
h_E or h	The Hermitian metric on E. For all smooth sections $u, v \in E$, we have $h_E(\theta u, v) = h_E(u, \varphi v)$.
$(E, \nabla, F^\bullet, Q)$	A complex polarization of Hodge structures
Some useful facts	• For each $E^{p,q}$, the connection $\nabla^{1,0} + \overline{\partial}$ is the metric connection with respect to h. • The curvature of the hermitian bundle $E^{p,q}$ is equal to $-(\theta\varphi + \varphi\theta)$.
Poincaré Type metric $\omega_{\text{caré}}$ on $(\Delta^*)^l \times \Delta^{n-l}$	$\omega_{\text{caré}} = i \sum_{j=1}^{l} \frac{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j}{\lvert z_j \rvert^2 (-\log \lvert z_j \rvert^2)^2} + \sum_{k=l+1}^{n} i\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$.
Admissible coordinates $(\Omega; z_1, \cdots, z_n)$ centered at $x \in D$	• Ω is an open subset of X containing x • $(z_1, \cdots, z_n)$ is a coordinates system on Ω centered at x, which gives a holomorphic isomorphism of Ω with $\Delta^n = \{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n : \lvert\zeta_j\rvert < 1\}$, • $D \cap \Omega$ is given by the equation $z_1 \cdots z_l = 0$ for some $l \leqslant n$.
Terminologies	***Interpretations***

Continued on next page

Table 3.1: Terminologies Interpretation (Continued)

Terminologies	Interpretations
Higgs field θ	If E is a variation of Hodge structures on $X = \overline{X}\backslash D$, then every point $x \in D$ has an admissible coordinate $(\Omega; z_1 \cdots, z_n)$ centered at x such that $\lvert\theta\rvert^2_{h,\text{caré}} \leqslant C$ holds on Ω^* for some constant $C > 0$.
Prolongation bundles via *Norm Growth and the Nilpotent Orbit Theorem*	Let (E, h) be a hermitian vector bundle on $X = \overline{X}\backslash D$. For each $\alpha = (\alpha_1, \cdots, \alpha_\nu) \in \mathbb{R}^n$, we prolong E to an $\mathcal{O}_{\overline{X}}$-module $\mathcal{P}_\alpha E$ as follows. Let $(U; z_1, \cdots, z_n)$ be an admissible coordinates and suppose that $D\|_U$ is defined by $z_1 \cdots z_l = 0$, i.e. $D_i = (z_i = 0), i = 1, \cdots, l$. Then $\mathcal{P}_\alpha E(U) = \left\{\sigma \in \Gamma(E, X \cap U) : \lvert\sigma\rvert \lesssim \prod_{j=1}^{l} \lvert z_j\rvert^{-\alpha_j-\varepsilon} \text{ on } U \text{ for all } \varepsilon > 0\right\}$. [a]
$D = \sum_{i=1}^{\nu} D_i$	A SNC divisor on a compact kähler manifold $\overline{X}$;
$\{\Omega_i\}_{i\in I}$	Finitely many admissible coordinates covering D s.t. $\lvert\theta\rvert^2_{h,\omega_{\text{caré}}} < C$;
$\mathcal{L}$	A line bundle on $\overline{X}$ s.t. $\mathcal{L} + D$ has a smooth hermitian metric $h_{\mathcal{L}+D} = h_{\mathcal{L}} + \sum_1^{\nu} \alpha_j h_j$ with **semi-positive curvature** $\omega_{\mathcal{L}+D} = \omega_{\mathcal{L}} + \sum_{j=1}^{\nu} \alpha_j \omega_j$ **and at each point** $x \in X$**, the curvature has at least** $n - t$ **positive eigenvalues;** (*)
B	a **nef** (semi-definite) line bundle on $\overline{X}$.(Hermitian metric: h_B ; Curvature w.r.t. h_B: ω_B)
Terminologies	***Interpretations***

[a] For every p, q and every $\alpha \in \mathbb{R}^\nu$, the prolongation bundles $\mathcal{P}_\alpha E^{p,q}$ satisfy the folllowing properties:
1. $\mathcal{P}_\alpha E^{p,q}$ is a locally free sheaf,
2. By Nilpotent Orbit theorem, $E^{p,q}$ can be naturally identified to $\mathcal{P}_\alpha E^{p,q}$, which is the prolongation bundle via growth of Higgs norm.

Theorem 3.1.1. (*Main theorem*)

Let $\mathcal{L}$ be a line bundle on $\overline{X}$. Assume that $\mathcal{L} + \sum_{i=1}^{\nu} \alpha_i D_i$ is $n - t$-positive $\mathbb{R}$-line bundle. If B is a nef line bundle on $\overline{X}$, then

$$\mathbb{H}^l\left(\overline{X}, \mathrm{gr}^p\left(\mathrm{DR}_{(\overline{X},D)}(E_\alpha)\right) \otimes \mathcal{L} \otimes B\right) = 0 \quad \text{for all } l > t, p \in \mathbb{Z}.$$

Remark. *This theorem is just the variant of [6, Theorem 1.1 (4.1)] by replacing $\Omega^p_X(\log D)$ with $\mathrm{gr}^p\left(\mathrm{DR}_{(\overline{X},D)}(E_\alpha)\right)$. Here we consider the Deligne extension $E_\alpha, \alpha \in \mathbb{R}^\nu$ of a variation of Hodge structures E on $\overline{X}\backslash D$, whose eigenvalues of the residue along D_i lie inside $[-\alpha_i, -\alpha_i + 1)$. By the Nilpotent Orbit theorem via Higgs norm growth, we gain the graded pieces $E^{p,q}_\alpha$ on $\overline{X}$ from extension of $E^{p,q}$. Then one has the graded pieces of the logarithmic de Rham complex $\mathrm{gr}^p\left(\mathrm{DR}_{(\overline{X},D)}(E_\alpha)\right)$ that is $(\Omega^i_{\overline{X}}(\log D) \otimes E^{p-r,q+r}_\alpha, \nabla)$, where $0 \leq i, r \leq n$, which is the generalization of [6, Theorem 3.1].*

Sec 3.2 Dolbeault Resolution for the de Rham Complex

3.2.1 The Dolbeault resolution

Let X be a complex manifold and $E \to X$ a holomorphic vector bundle. Let $\mathcal{E}$ be the associated sheaf of free $\mathcal{O}_X$-modules. Let $\mathcal{A}^{0,q}(E)$ be the sheaf of $\mathcal{C}^\infty$ sections of $\Omega^{0,q} \otimes E$. In (2.5), we defined the operator

$$\bar{\partial} : \mathcal{A}^{0,q}(E) \to \mathcal{A}^{0,q+1}(E).$$

We know (cf. lemma 2.34 and proposition 2.36) that this operator satisfies:

- The kernel of $\bar{\partial} : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,1}(E)$ is equal to the sheaf of holomorphic sections of E, i.e. to $\mathcal{E}$ (here $\mathcal{A}^{0,0}(E)$ is the sheaf of $\mathcal{C}^\infty$ sections of E).
- For $q > 0$, a section of $\mathcal{A}^{0,q}(E)$ is $\bar{\partial}$-closed if and only if it is locally $\bar{\partial}$-exact.

In other words, we have the following.

Proposition 3.2.1.

The complex

$$0 \to \mathcal{A}^{0,0}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(E) \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{0,n}(E) \to 0,$$

where $n = \dim_{\mathbb{C}} X$, is a resolution of the sheaf $\mathcal{E}$.

3.2.2 The logarithmic Holomorphic de Rham Resolution

Let X be a complex manifold, and let $D \subset X$ be a hypersurface, i.e. D is locally defined by the vanishing of a holomorphic equation.

Definition 3.2.1. (*Normal Crossing Divisor*)

We say that D is a *normal crossing divisor* if locally there exist coordinates $z_1, \ldots, z_n$ on X such that D is defined by the equation $z_1 \cdots z_r = 0$ for an integer r which naturally depends on the considered open set.
In particular, a divisor $D = \sum_{i=1}^{s} D_i$ is called *simple normal crossing divisor* if every irreducible component D_i is smooth and all intersections are *transverse*.

Given a pair (X, D), where D is a normal crossing divisor in X, we will define the holomorphic de Rham complex with logarithmic singularities along D. Let $\Omega_X^k(\log D)$ be the subsheaf of the sheaf $\Omega_X^k(*D)$ of meromorphic forms on X, holomorphic on $X - D$, defined by the condition:

- If α is a meromorphic differential form on U, holomorphic on $U - D \cap U$, $\alpha \in \Omega_X^k(\log D)_{|U}$ if α admits a pole of order at most 1 along (each component of) D, and the same holds for $d\alpha$.

Lemma 3.2.1.

Let $z_1, \ldots, z_n$ be local coordinates on an open set U of X, in which $D \cap U$ is defined by the equation $z_1 \cdots z_r = 0$. Then $\Omega^k_X(\log D)_{|U}$ is a sheaf of free $\mathcal{O}_U$-modules, for which $\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_1}}{z_{i_l}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_m}$ with $i_s \leq r$, $j_s > r$ and $l + m = k$ form a basis.

***Proof*.** Let α be a section of $\Omega^k_X(\log D)$ on $V \subset U$. As α admits a pole of order at most 1 along D, we can write $\alpha = \frac{\beta}{z_1 \cdots z_r}$, with β a holomorphic k-form on V. As $d\alpha$ admits a pole of order at most 1 along D, we find that $\sum_{i \leq r} z_1 \cdots \hat{z}_i \cdots\ z_r dz_i \wedge \beta$ must vanish along D. It follows immediately that if $\beta = \sum_{I,J} \beta_{I,J} dz_I \wedge dz_J$ with $I \subset \{1, \ldots, r\}$, $J \subset \{r+1, \ldots, n\}$, the function $\beta_{I,J}$ must vanish on the hyperplanes of equation $z_i, i \in I' := \{1, \ldots, r\} - I$, and thus must be divisible by $z_{I'} = \prod_{i \in I'} z_i$. ■

Corollary 3.2.1.

The sheaves $\Omega^k_X(\log D)$ are sheaves of free $\mathcal{O}_X$-modules.

Furthermore, by definition, if α is a section of $\Omega^k_X(\log D)$ on $V \subset X$, then $d\alpha = \partial\alpha$ is in $\Omega^{k+1}_X(\log D)$. Indeed $d\alpha$ is meromorphic, with a pole of order at most 1 along D, and closed. Thus, $(\Omega_X(\log D), \partial)$ is a complex of sheaves over X. This complex is called *the logarithmic de Rham complex*.

For **Corollary 3.2.1** , the complex

$$0 \to \mathcal{A}^{0,0}_X(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}_X(E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{0,k}_X(E) \to 0$$

is a resolution for $\Omega^k_X(\log D)$, where $\mathcal{A}^{0,k}_X(E) = \bigoplus_{p+q=k} (\Omega^k_X \otimes E^{p,q})$.

Lemma 3.2.2.

Via i, the holomorphic de Rham complex is a resolution of $\mathbb{C}$.

***Proof*.** We want to show that the sheaves of cohomology $\mathcal{H}^k = \mathcal{H}^k(\Omega_X)$ satisfy $\mathcal{H}^0 = i(\mathbb{C})$ and $\mathcal{H}^k = 0$ for $k > 0$.

Now, we have an inclusion of the holomorphic de Rham complex into the de Rham complex

$$(\Omega_X, \partial) \to (\mathcal{A}^k_X, d)\,,$$

since d and ∂ coincide on holomorphic forms. Moreover, we can see the usual de Rham complex $\mathcal{A}_X$ as the simple complex associated to the double complex

$$(\mathcal{A}^{p,q}, \partial, (-1)^p \bar{\partial})\,.$$

Each column $(\mathcal{A}^{p,q}_X, (-1)^p \bar{\partial})$ of this double complex is exact in positive degree by proposition 2.36 and gives a resolution of Ω^p_X. Thus, the de Rham complex is quasi-isomorphic to the holomorphic de Rham complex by lemma 8.5. Like the usual de Rham complex, it is exact in positive degree, and its cohomology is given by the locally constant sheaf $\mathbb{C}$ in degree 0 , so this also holds for the holomorphic de Rham complex. ■

Sec 3.3 L^2-existence results

Lemma 3.3.3.

Let H_1, H_2 and H_3 be Hilbert spaces and let $T\colon H_1 \to H_2$ and $S\colon H_2 \to H_3$ be closed and densely defined operators such that $ST = 0$. Let T^* and S^* be the adjoints of T and S, respectively. Suppose that there exists $\varepsilon > 0$ such that

$$\|T^*u\|^2 + \|Su\|^2 \geq \varepsilon^2\|u\|^2 \qquad \text{for all } u \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S).$$

Then for every $u \in H_2$ such that $Su = 0$, there exists $v \in H_1$ such that $Tv = u$ and $\|v\| \leq \varepsilon^{-1}\|u\|$.

The first proposition is a global version of the L^2-existence result in **Lemma 3.3.3**.

Theorem 3.3.2. ***(Global version of the L^2-existence result)***

Let (X, ω) be a Kähler manifold (not necessarily compact) and let $\mathcal{L}$ be a line bundle with smooth hermitian metric $h_{\mathcal{L}}$. Let E be a complex polarized variation of Hodge structures on X. Furthermore, we assume that

1. *The geodesic distance δ_ω is complete on X.*
2. *The norm of the Higgs field $|\theta|^2_{h,\omega}$ is globally bounded.*
3. *There exists $\varepsilon > 0$ such that $\langle [i\Theta_{\mathcal{L}}, \Lambda_\omega]u, u\rangle \geq \varepsilon\|u\|^2_\omega$ for all smooth and compactly supported (r, s)-forms u for $r + s = n + l$.*

Let $\boldsymbol{u}$ be a measurable section with values in $\mathbb{E}^l$ such that $\eth\boldsymbol{u} = 0$. Provided that the right hand side of the expression below is finite, there exists a measurable section $\boldsymbol{v}$ with values in $\mathbb{E}^{l-1}$ satisfying $\eth\boldsymbol{v} = \boldsymbol{u}$ and the following inequality

$$\int_X \|\boldsymbol{v}\|^2_{h,\omega} dV_\omega \leqslant \frac{1}{\varepsilon}\int_X \|\boldsymbol{u}\|^2_{h,\omega} dV_\omega.$$

Theorem 3.3.3. ***(Local version of the L^2-existence result)***

Equip $\Omega^* = (\Delta^*)^l \times \Delta^{n-l}$ with the Poincaré metric $\omega_{car\acute{e}}$ defined in Table 3.1. Let E be a complex polarized variation of Hodge structures on Ω^*. Let $C > 0$ be a number such that $\|\theta\|^2_{\omega_{car\acute{e}}} < C$ on Ω^*. Define $\eta\colon \Omega^* \to \mathbb{R}$ by the following formula, for $a_j \in \mathbb{R}$ and $b_j > C + 2$:

$$e^{-\eta} = \prod_{j=1}^{l} |z_j|^{2a_j}(-\log|z_j|^2)^{b_j} \prod_{j=l+1}^{n} e^{-b_j|z_j|^2}.$$

If u is an (r, s)-form with values in $E^{p,q}$, with measurable coefficients such that $\bar{\partial}u = 0$ and

$$\int_{\Omega^*} \|u\|^2_{h,\omega_{car\acute{e}}} e^{-\eta} dV_{car\acute{e}} < +\infty,$$

then there exists an $(r, s-1)$-form with values in $E^{p,q}$, with measurable coefficients, such that $u = \bar{\partial}v$ and

$$\int_{\Omega^*} \|v\|^2_{h,\omega_{car\acute{e}}} e^{-\eta} dV_{car\acute{e}} \leq \int_{\Omega^*} \|u\|^2_{h,\omega_{car\acute{e}}} e^{-\eta} dV_{car\acute{e}}.$$

Sec 3.4 Construction of the Kähler metric on the Complement

1. As $\overline{X}$ is a Kähler manifold, let ω_0 be the Kähler metric on $\overline{X}$. (positive-definite)
2. According to (*), we can choose hermitian metrics on $\mathcal{L}$ and $\mathcal{O}_{\overline{X}}(D_j)$, and abtain the corresponding curvatures $\omega_{\mathcal{L}}$ and $\{\omega_j\}_1^\nu$. Locally, for each $x \in X$, we can simulataneously diagonalize ω_0 and $\omega_{\mathcal{L}} + \sum_{j=1}^{\nu} \alpha_j D_j$ such that
$$\omega_0(x) = i \sum_{\mu} \zeta_\mu \wedge \overline{\zeta_\mu} \qquad \& \qquad \omega_{\mathcal{L}} + \sum_{j=1}^{\nu} \alpha_j \omega_j = i \sum_{\mu} \gamma_\mu(x) \zeta_\mu \wedge \overline{\zeta_\mu}.$$
(diagonalization of the metric and corresponding curvature), where $0 \leqslant \gamma_1(x) \leqslant \cdots \leqslant \gamma_n(x)$ and $\gamma_t(x) > 0$ since there at least $n-t$ positive eigenvalues. We can let $m = \min_{x \in \overline{X}} \gamma_t(x) > 0$.
3. (Constructing the properiate Kähler metric ω from ω_0. How and Why?) From the proof of [Demailly,ChapterVII,4]: "Let us consider the new Kahler metric on X
$$\omega_\varepsilon = \varepsilon\omega + i\Theta(E), \varepsilon > 0,$$
and let $i\Theta(E) = i \sum \gamma_j \zeta_j \wedge \overline{\zeta_j}$ be a diagonalization of $i\Theta(E)$ with respect to ω and with $\gamma_1 \leqslant \cdots \leqslant \gamma_n$. Then $\omega_\varepsilon = i \sum (\varepsilon + \gamma_j) \zeta_j \wedge \overline{\zeta_j}$. The eigenvalues of $i\Theta(E)$ with respect to ω_ε are given by
$$\gamma_{j,\varepsilon} = \frac{\gamma_j}{(\varepsilon + \gamma_j)}, 1 \leqslant j \leqslant n.$$
",we know that we can construct an analogous Kähler metric such that it is complete for each admissible coordinate $\Omega_i, i \in I$.
4. *Let $\sigma_j \in H^0\left(\overline{X}, \mathcal{O}_{\overline{X}}(D_j)\right) = \Gamma\left(\overline{X}, \mathcal{O}_{\overline{X}}(D_j)\right)$ that vanishes along D_j.*
$$\sigma_j(x) = \begin{cases} 0, & x \in D_j; \\ \neq 0, & x \in \overline{X} \backslash D_j. \end{cases}$$
And take a_j to be sufficiently close to α_j so that $\mathcal{P}_\alpha(E^{p,q}) = \mathcal{P}_a(E^{p,q})$. We can rescale σ_j so that $|\sigma_j|^2_{h_j} \leqslant \exp\left(\frac{-2(C+3)}{\delta\varepsilon_2}\right)$. Let
$$\tau_j : X \to \mathbb{R}, \quad \tau_j(x) = -\log|\sigma_j|^2_{h_j,x}$$
and let
$$\xi = \sum_{j=1}^{\nu} a_j \tau_j - b_j \log \tau_j.$$
So define the new Kähler metric ω on X as
$$\omega = \varepsilon\omega_0 + \omega_{\mathcal{L}} + \omega_B + i\partial\overline{\partial}\xi.$$
ω is positive definite and then ω is a Kähler form on X .
5. (Prove that the metric ω and $\omega_{\text{caré}}$ are mutually bounded for each admisiible coordinate Ω_i.) ① Clearly, this implies that (X, ω) is complete. Fix an admisiible coordinate $(\Omega; z_1, \cdots, z_n)$

and assume that $D \cap \Omega$ is defined by the equation $z_1 \cdots z_l = 0$. For convenience, suppose that $(z_i = 0) = D_i \cap \Omega$. Note that $\tau_i = -\log|z_i|^2 + g_i$ for some smooth function g_i on Ω. Then

$$\frac{b_j}{\tau_j^2}\partial\tau_j \wedge \overline{\partial}\tau_j = \frac{b_j}{(-\log|z_j|^2+g)^2}\left(\frac{\mathrm{d}z_j}{z_j} - \partial g_j\right) \wedge \left(\frac{\mathrm{d}\overline{z}_j}{\overline{z}_j} - \overline{\partial} g_j\right).$$

Since the first four terms of ω are smooth on $\overline{X}$, we see that ② ω and $\omega_{\text{caré}}$ are mutually bounded on Ω. ③ This also shows that $|\theta|^2_{h_j,\omega}$ is globally bounded (Theorem 2.5[4]).

3.4.1 Poincaré type Kähler metric on complement $X \backslash D$

Namely[1, Introduction, Definition 2], fixing a simple normal crossing divisor D in a compact Kähler manifold (X, J, ω_X), we recall the definition Poincaré type Kähler metrics on $X \backslash D$, following [TY87, Wu08, Auv11]:

Definition 3.4.2. ***(Poincaré type Kähler metric on complement)***

A smooth positive $(1,1)$-form ω on $X \backslash D$ is called a Poincaré type Kähler metric on $X \backslash D$ if: on every open subset U of coordinates $(z^1, \dots, z^m)$ in X, in which D is given by $\{z^1 \cdots z^j = 0\}$, ω is mutually bounded with ②

$$(\omega_{\text{caré}})\ \omega_U^{\text{model}} := i\sum_{k=1}^{j} \frac{\mathrm{d}z^k \wedge \mathrm{d}\overline{z^k}}{|z^k|^2 \log^2\left(|z^k|^2\right)} + i\sum_{l=j+1}^{m} \mathrm{d}z^l \wedge \mathrm{d}\overline{z^l}$$

and has bounded derivatives at any order for this model metric.
We say moreover that ω is of class $[\omega_X]$ if $\omega = \omega_X + dd^c\varphi$ for some φ smooth on $X \backslash D$, with $\varphi = \mathcal{O}\left(\sum_{\ell=1}^{j} \log\left[-\log\left(|z^\ell|^2\right)\right]\right)$ in the above coordinates and $d\varphi$ bounded at any order for ω_U^{mdl}. We then set: $\omega \in \mathscr{M}^D_{[\omega_X]}$.

① Metrics of $\mathscr{M}^D_{[\omega_X]}$ are complete, with finite volume (equal to that of X for smooth Kähler metrics of class $[\omega_X]$); they also share a common mean scalar curvature, which differs from that attached to smooth Kähler metrics of class $[\omega_X]$.

3.4.2 Check the positivity of the commutator operator $[i\Theta_{\mathcal{L}\otimes B}, \Lambda_\omega]$

The key idea is to *twist the metric of $\mathcal{L}$ by an extra factor of e^ξ*. This gives us a smooth hermitian metric $h'_{\mathcal{L}} = h_{\mathcal{L}}e^\xi$ on $\mathcal{L}|_X$. Denote it by $\Theta'_{\mathcal{L}\otimes B}$. Then

$$i\Theta'_{\mathcal{L}\otimes B} = \omega_{\mathcal{L}} + \omega_B + i\partial\overline{\partial}\xi.$$

simulataneously diagonalization ω_0 and $i\Theta'_{\mathcal{L}\otimes B}$ at $x \in X$ and express

$$\omega_0 = i\sum_\mu \zeta_\mu \wedge \overline{\zeta_\mu} \qquad \& \qquad i\Theta'_{\mathcal{L}\otimes B} = i\sum_\mu \gamma'_\mu(x)\zeta_\mu \wedge \overline{\zeta_\mu},$$

[4]Which is related to the existence of admissible coordinates centered at each point $x \in D$.

and $\gamma_1' \leqslant \cdots \leqslant \gamma_n'$.

Using the result of 3. of ***The sketch of the proof*** and $\omega = \varepsilon\omega_0 + i\Theta'_{\mathcal{L}\otimes B}$, if we diagonalize $i\Theta'_{\mathcal{L}\otimes B}$ with respect to ω and denote the eigenvalues as $\gamma'_{\mu,\varepsilon}(x)$, then we have

$$\gamma'_{\mu,\varepsilon} = \frac{\gamma'_\mu}{\gamma'_\mu + \varepsilon}.$$

As

$$\begin{aligned} 1 \geqslant \gamma'_{\mu,\varepsilon}(x) \geqslant \frac{\varepsilon_1 - \varepsilon}{\varepsilon_1} \quad & \text{for } 1 \leq \mu \leq t-1, \\ 1 \geqslant \gamma'_{\mu,\varepsilon}(x) \geqslant \frac{m + \varepsilon_1 - \varepsilon}{m + \varepsilon_1} \quad & \text{for } t \leq \mu \leq n. \end{aligned}$$

,then we abtain that [8]

$$\begin{aligned} \langle [i\Theta_{\mathcal{L}\otimes B}, \Lambda_\omega]u, u\rangle_x &\overset{\text{lemma2.12}}{\geqslant} \left(\gamma'_{1,\varepsilon}(x) + \cdots + \gamma'_{s,\varepsilon}(x) - \gamma'_{r+1,\varepsilon}(x) - \cdots - \gamma'_{n,\varepsilon}(x)\right) |u|^2_{\omega,x} \\ &\geqslant \left[\left(\frac{\varepsilon_1 - \varepsilon}{\varepsilon_1}\right) \cdot t + \left(\frac{m + \varepsilon_1 - \varepsilon}{m + \varepsilon_1}\right) \cdot (s - t) - (n - r)\right] |u|^2_{\omega,x} \\ &\geqslant \frac{1}{10} |u|^2_{\omega,x}. \end{aligned}$$

Sec 3.5 L^2–Dolbeault resolution

3.5.1 L^2–Dolbeault resolution of the de Rham complex when there is a variation of Hodge structures on the complement of an SNC devisor

TARGET

Construct a Kähler metric ω on X s.t. the following conditions are satisfied:

1. (X, ω) *is* ***complete****;*
2. $|\theta|^2_{h,\omega_{\text{care}}}$ *is* ***globally bounded****;*
3. $[i\Theta_{\mathcal{L}\otimes B}, \Lambda_\omega]$ *is a* ***positive definite*** (r, s)*-form;*
4. *The local $\bar\partial$-equation in admissible coordinate with appropriate twist is* ***solvable****.*

Proposition 3.5.2.

The complex

$$0 \to \mathcal{H}^2_{(r)}(E^{p,q} \otimes \mathcal{L} \otimes B) \to L^2_{(r,\bullet)}(E^{p,q} \otimes \mathcal{L} \otimes B)$$

is a resolution of $\mathcal{H}^2_{(r)}(E^{p,q} \otimes \mathcal{L} \otimes B)$ by fine sheaves.

Proof. By the short exact sequence

$$\mathcal{H}^2_{(r)}(E^{p,q}\otimes\mathcal{L}\otimes B)\xrightarrow{i}L^2_{(r,0)}(E^{p,q}\otimes\mathcal{L}\otimes B)\xrightarrow{\overline{\partial}}L^2_{(r,1)}(E^{p,q}\otimes\mathcal{L}\otimes B)$$

and i being injective with $\mathrm{Im}(i)=\ker\overline{\partial}$, we know that the complex is exact at $\mathcal{H}^2_{(r)}(E^{p,q}\otimes\mathcal{L}\otimes B)$. (The $\overline{\partial}$–equation is regular.) Also, it is clear that the complex is exact on X. *Thus, the left task is to prove the exactness on the boundary $\overline{X}$, which is equivalent to solve a $\overline{\partial}$-equation on a doamin of type $\Omega^*=(\Delta^*)^l\times\Delta^{n-l}$.*

After construction of an admissible coodinate $(\Omega;z_1,\cdots,z_n)$ $(D=(z_1\cdots z_l=0))$ and assuming that $\mathcal{L}\otimes B$ is locally trivial on Ω and is trivialized by a non-vanishing section σ, let $u\otimes\sigma\in L^2_{(r,\bullet)}(E^{p,q}\otimes\mathcal{L}\otimes B)(\Omega)$ such that $\overline{\partial}u=0$ with

$$\int_{\Omega^*}\|u\|^2_{h_E,\omega_{\text{car\'e}}}e^{-\eta}\mathrm{d}V_{\omega_{\text{car\'e}}}<+\infty,$$

then by proposition 3.5, the $\overline{\partial}$-equation is solvable. ■

Proposition 3.5.3.

The sheaves $\mathcal{H}^2_{(r)}(E^{p,q}\otimes\mathcal{L}\otimes B)$ in terms of prolongation bundles are

$$\mathcal{H}^2_{(r)}(E^{p,q}\otimes\mathcal{L}\otimes B)=\Omega^r_{\overline{X}}(\log D)\otimes\mathcal{P}_\alpha E^{p,q}\otimes\mathcal{L}\otimes B.$$

Proof. ■

Sec 3.6 Proof of the Main theorem

1. As the global bound for the Higgs field θ in Theorem 2.5 give a morphism on sheaves $L^2_{(r,s)}(E^{p,q}\otimes\mathcal{L}\otimes B)\xrightarrow{\theta}L^2_{(r+1,s)}(E^{p,q}\otimes\mathcal{L}\otimes B)$, thus analogously to Section 3.1, we can construct a double complex:

$$\begin{array}{ccccccc}
L^2_{(0,n)}(E^{p,q}\otimes\mathcal{L}\otimes B) & \xrightarrow{\theta} & L^2_{(1,n)}(E^{p-1,q+1}\otimes\mathcal{L}\otimes B) & \xrightarrow{\theta} & \cdots & \xrightarrow{\theta} & L^2_{(n,n)}(E^{p-n,q+n}\otimes\mathcal{L}\otimes B)\\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow\\
\vdots & \xrightarrow{\theta} & \vdots & \xrightarrow{\theta} & \vdots & \xrightarrow{\theta} & \vdots\\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow\\
L^2_{(0,1)}(E^{p,q}\otimes\mathcal{L}\otimes B) & \xrightarrow{\theta} & L^2_{(1,1)}(E^{p-1,q+1}\otimes\mathcal{L}\otimes B) & \xrightarrow{\theta} & \cdots & \xrightarrow{\theta} & L^2_{(1,n)}(E^{p-n,q+n}\otimes\mathcal{L}\otimes B)\\
\overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow & & \overline{\partial}\uparrow\\
L^2_{(0,0)}(E^{p,q}\otimes\mathcal{L}\otimes B) & \xrightarrow{\theta} & L^2_{(1,0)}(E^{p-1,q+1}\otimes\mathcal{L}\otimes B) & \xrightarrow{\theta} & \cdots & \xrightarrow{\theta} & L^2_{(n,0)}(E^{p-n,q+n}\otimes\mathcal{L}\otimes B).
\end{array}$$

2. By Proposition 3.8 and 3.9, the r-th column is a resolution of $\Omega^r_{\overline{X}}(\log D)\otimes E^{p-r,q+r}_\alpha\otimes\mathcal{L}\otimes B$ by fine sheaves. Hence, we can compute the hypercohomology of

$$\left[E^{p,q}_\alpha\to\Omega^1_{\overline{X}}(\log D)\otimes E^{p-1,q+1}_\alpha\to\cdots\to\Omega^n_{\overline{X}}(\log D)\otimes E^{p-n,q+n}_\alpha\right][n]\otimes\mathcal{L}\otimes B$$

by taking the global section of the total complex above and compute the cohomology. Let $(\mathbb{E},\eth)$ be the total complex of the double complex above. We have a concrete description of the global section of the sheaves $L^2_{(r,s)}(E^{p-r,q+r} \otimes \mathcal{L} \otimes B)$. The global sections u are (r, s)-forms on X with values in $E^{p-r,q+r} \otimes \mathcal{L} \otimes B$ with measurable coefficients such that

$$\int_X \|u\|^2_{h',\omega} dV_\omega < +\infty \qquad \text{and} \qquad \int_X \left\|\bar{\partial}u\right\|^2_{h',\omega} dV_\omega < +\infty.$$

By Proposition 3.6 and 3.3, we have an a priori inequality (***The condition ③ of prop 3.4***)

$$\|\eth\mathbf{u}\|^2 + \|\eth^*\mathbf{u}\|^2 \geq 0.1\|\mathbf{u}\|^2$$

for $\mathbf{u} \in \mathbb{E}^l$ when $l > 0$. ***Since (X, ω) is complete and $|\theta|^2_{h_E,\omega}$ is globally bounded on X***, the vanishing of cohomology immediately follows from ***Proposition 3.4***.

LOGARITHIMIC VANISHING THEOREMS FOR EFFECTIVE q-AMPLE DIVISORS

4

Part I

Sec 4.1 Introduction

Definition 4.1.1. (*q-ample line bundle*)

A line bundle L over a compact complex manifold X is called q-ample if for any *coherent sheaf* $\mathcal{F}$ on X there exists a positive integer $m_0 = m_0(X, L, \mathcal{F}) > 0$ such that

$$H^i(X, \mathcal{F} \otimes L^m) = 0, \text{ for } i > q, m \geqslant m_0.$$

Remark. *A divisor D is q-ample if $\mathcal{O}_X(D)$ is a q-ample line bundle.*

Note that for $\Delta = \sum_{i=1}^{s} \alpha_i D_i \in \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, $\alpha_i \in \mathbb{R}$ to be a q-positive (q-ample) $\mathbb{R}$-divisor, Δ is the support of some q-ample divisor D', i.e. $\mathrm{Supp}(D') = \Delta$.

Theorem 4.1.1. (*Main theorem*)

Let

X	*A compact Kähler manifold*
D	*A small normal crossing divisor which is the support of some effective q-ample divisor D', i.e.* $\mathrm{Supp}(D') = D$.

Then we have

$$H^i(X, \Omega_X^j(\log D)) = 0, \text{ for any } i + j > n + q.$$

Theorem 4.1.2. (*generalization of Main theorem*)

With the same notation above, for any nef line bundle L, we have

$$H^i(X, \Omega_X^j(\log D) \otimes L) = 0, \text{ for any } i + j > n + q + 1.$$

If $H(t,x) = tx$ *is the homotopy between the identity map* $\Omega \to \Omega$ *and the constant map* $\Omega \to \{0\}$. *Then we have the following computation:*

$$\begin{cases} F(x) = H(0,x) \equiv 0 \\ G(x) = H(1,x) \equiv \mathrm{Id}_\Omega \,. \end{cases}$$

and

$$\begin{cases} F^*(v) = \begin{cases} v(0) \in H^0_{DR}(\Omega, \mathbb{R}) = \mathbb{R}; \\ 0 \in H^p_{DR}(\Omega, \mathbb{R}) = \{0\}.(\textit{For } \mathrm{d}^p(v(0)) \equiv 0 \textit{ for any } p \geq 1) \end{cases} \\ G^*(v) = (\mathrm{Id})^*(v) = v. \end{cases},$$

5

Notes for some new topics

Part I

Sec 5.1 Perverse Sheaf and Intersection Cohomology

5.1.1 Poincaré Duality

Definition 5.1.1. ***(cap product)***

On an n-manifold X, the cap product is

$$C^i(X) \times C_n(X) \xrightarrow{\frown} C_{n-i}(X),$$

where C_i and C^i denote the (simplicial/singular) i-(co)chains on X with $\mathbb{Z}_{-}$ coefficients.

The cap product is defined as follows: if $a \in C^{n-i}(X), b \in C^i(X)$ and $\sigma \in C_n(X)$, then

$$a(b \frown \sigma) = (a \smile b)(\sigma).$$

The cap product is compatible with the boundary maps, thus it descends to a map

$$H^i(X;\mathbb{Z}) \times H_n(X;\mathbb{Z}) \xrightarrow{\frown} H_{n-i}(X;\mathbb{Z}).$$

The following statement lies at the heart of algebraic and geometric topology. For a modern proof see, e.g., [5, Section 3.3]:

Theorem 5.1.1. ***(Poincaré Duality)***

Let X be a closed, connected, oriented topological n-manifold with fundamental class $[X]$. Then capping with $[X]$ gives an isomorphism

$$H^i(X;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(X;\mathbb{Z})$$

for all integers i.

As a consequence of Theorem 5.1.1 one gets a non-degenerate pairing

$$H_i(X,\mathbb{C}) \otimes H_{n-i}(X;\mathbb{C}) \longrightarrow \mathbb{C}.$$

In particular, the Betti numbers (It is known as the rank of the corresponding homology groups.) of X in complementary degrees coincide, i.e.,

$$\dim_{\mathbb{C}} H_i(X;\mathbb{C}) = \dim_{\mathbb{C}} H_{n-i}(X;\mathbb{C}).$$

Note that the existence of Hodge structures on the cohomology of complex projective manifolds leads to an important consequence that the odd Betti numbers of a complex projective manifold are even.

5.1.2 Understanding Why the Odd Betti Numbers of a Complex Projective Manifold are Even?

For a complex projective manifold, the odd Betti numbers are always even. This can be understood through a combination of complex geometry and topological properties. Let's break this down in detail:

1. **Definition of Betti Numbers:** Betti numbers, denoted as b_k, quantify the topology of a manifold by representing the rank of the k-th homology group $H_k(M, \mathbb{Z})$ (or the k-th cohomology group $H^k(X; \mathbb{Z})$). They indicate the number of k-dimensional "holes" or independent cycles in the manifold. For instance, b_0 represents the number of connected components, b_1 represents the number of independent loops, and so on.
2. **Complex Projective Manifolds:** A complex projective manifold is a complex manifold that can be embedded into complex projective space. These manifolds have a rich structure and are inherently Kähler manifolds, meaning they have a compatible triple structure of a complex structure, a symplectic structure, and a Riemannian metric.
3. **Hodge Decomposition:** For a Kähler manifold M, the complex de Rham cohomology group $H^k(M, \mathbb{C})$ can be decomposed into a direct sum of Hodge components:

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$$

   Here, $H^{p,q}(M)$ denotes the space of harmonic forms of type (p, q), and $h^{p,q} = \dim H^{p,q}(M)$ are the Hodge numbers.
4. **Relation Between Betti Numbers and Hodge Numbers:** The k-th Betti number b_k is related to the Hodge numbers $h^{p,q}$ by the following formula:

$$b_k = \sum_{p+q=k} h^{p,q}$$

5. **Symmetry of Hodge Numbers:** For Kähler manifolds, there is a fundamental symmetry in the Hodge numbers:

$$h^{p,q} = h^{q,p}$$

   This symmetry implies that the Hodge components $H^{p,q}$ and $H^{q,p}$ appear in pairs.
6. **Implication for Odd Betti Numbers:** Due to the symmetry $h^{p,q} = h^{q,p}$, the sum of Hodge numbers for odd k (such as b_1, b_3, etc.) will always be an even number because each non-zero $h^{p,q}$ has a matching $h^{q,p}$. Thus, the odd Betti numbers must be even.
7. **Example:** Consider the complex projective space $\mathbb{CP}^n$. The Hodge numbers are as follows:
   - $h^{0,0} = 1$
   - $h^{1,1} = 1$
   - $h^{2,2} = 1$ (if $n \geq 2$)
   - All other $h^{p,q} = 0$.

   The Betti numbers calculated are:
   - $b_0 = h^{0,0} = 1$
   - $b_2 = h^{1,1} = 1$
   - $b_4 = h^{2,2} = 1$ (for $n \geq 2$)
   - The odd Betti numbers $b_1 = b_3 = 0$.

This example shows that odd Betti numbers are zero (which is even) for $\mathbb{CP}^n$.

8. **Conclusion:** In summary, the reason the odd Betti numbers of a complex projective manifold are even is due to the Hodge decomposition and the inherent symmetry of Hodge numbers on Kähler manifolds.

Remark. In the diagram, δ is labeled as a **meridian**, and η is labeled as a **longitude**. The reason why the homology class of δ vanishes can be explained from the perspective of algebraic topology.

1. **Meridian as a Boundary**: From the diagram, the meridian δ appears to be the boundary of a region. In homology theory, any curve that forms the boundary of a region has a **trivial homology class** (i.e., it vanishes). This is because a boundary does not represent a closed, independent cycle—it is merely the edge of a higher-dimensional region. In other words, since δ bounds some region within X, it is a boundary, and hence its homology class must vanish.

2. **Boundaries and Homology in Algebraic Topology**: In homology theory, the boundary of a higher-dimensional object always has a zero homology class. For example, in the case of a surface, if a loop (like the meridian δ) is the boundary of a region, its homology class is trivial because it does not represent a free, closed cycle but rather a boundary.

3. **Betti Number and Hodge Decomposition**: The passage also mentions that δ's homology class vanishes, and this is related to the fact that the first Betti number b_1 of X is odd. According to Hodge theory, if the first Betti number is odd, a complete Hodge decomposition cannot exist. This implies that certain homology classes in $H^1(X;\mathbb{C})$ cannot be fully decomposed into pure $(1,0)$ and $(0,1)$ components. This is connected to the fact that δ's homology class vanishes in the homology of X.

Summary:

1. The meridian δ is the boundary of some region, and by the fundamental property of homology, **any boundary has a trivial homology class**.
2. This follows from basic algebraic topology, where boundaries do not contribute to non-trivial homology classes.
3. Additionally, the fact that X has an odd first Betti number implies that a full Hodge decomposition is not possible for $H^1(X;\mathbb{C})$, which further supports why the homology class of δ is trivial.

5.1.3 Lefschetz Hyperplane Section Theorem

A map $f: X \to Y$ is called **homotopy equivalence** if there is a map $g: Y \to X$ such that $fg \cong \mathbb{I}$ and $gf \cong \mathbb{I}$. It is an equivalent relation and X and Y are homotopy equivalent if they are the deformation retracts of the third space Z containing them. In general, we can take Z as the mapping Cylinder M_f of any homotopy equivalence $f: X \to Y$. As we know that M_f deformation retracts to Y, it suffices to prove that M_f also deformation retracts to its other end X.

BIBLIOGRAPHY

[1] Hugues Auvray. "Note on Poincaré type Kähler metrics and Futaki characters". In: *arXiv preprint arXiv:1401.0128* (2013).

[2] Eugenio Calabi and Edoardo Vesentini. "On compact, locally symmetric Kahler manifolds". In: *Annals of Mathematics* (1960), pp. 472–507.

[3] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry*. 1st ed. Vol. 1. Book. book. Institut Fourier, UMR 5582 du CNRS: Université de Grenoble I, June 2012.

[4] Giuliana Gigante. "Vector bundles with semidefinite curvature and cohomology vanishing theorems". In: *Adv. in Math* 41.1 (1981), pp. 40–56.

[5] Allen Hatcher. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002.

[6] Chunle Huang et al. "Logarithmic vanishing theorems on compact Kähler manifolds I". In: *arXiv preprint arXiv:1611.07671* (2016).

[7] Huybrechts and Daniel. *Complex geometry:an introduction*. Springer, 2010.

[8] Hyunsuk Kim. "L^2-approach to the Saito vanishing theorem". In: *arXiv preprint arXiv: 2306.00313* (2023).

[9] Shoshichi Kobayashi and Takushiro Ochiai. "On complex manifolds with positive tangent bundles". In: *Journal of the Mathematical Society of Japan* 22.4 (1970), pp. 499–525.

[10] Edoardo Vesentini and MS Raghunathan. *Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems*. Vol. 39. Tata Institute of fundamental research Bombay, 1967.

[11] Steven Zucker. "Hodge theory with degenerating coefficients: L^2 cohomology in the Poincaré metric". In: *Annals of Mathematics* (1979), pp. 415–476.

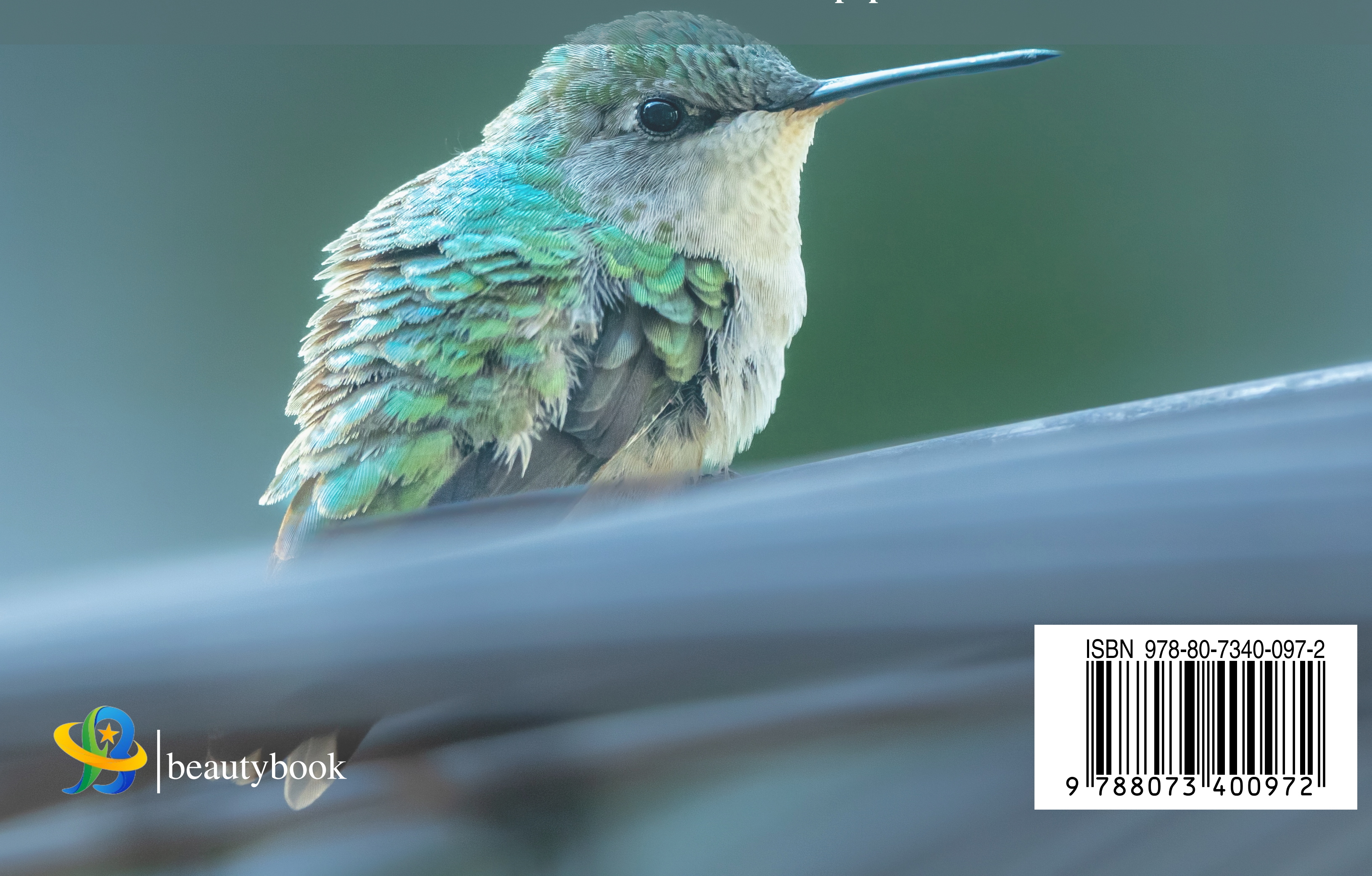
INDEX

Preliminaries, 3

Vector Bundles with Semidefinite Curvature and Cohomology Vanishing Theorems, 3

RESEARCH NOTES

A Research Notes Series For papers.



beautybook

ISBN 978-80-7340-097-2

9 788073 400972