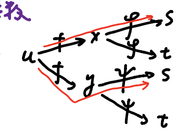
第十一章 多元函数微分学

§11.1 偏导数

链式法则：



则 $\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$

f'_{xy} 与 f'_{yx}：可能不等. 如 $f(x,y)=\begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & x^2+y^2\neq 0 \\ 0 & x^2+y^2=0 \end{cases}$

f'_x, f'_y 在某点的邻域内存在 且 f'_{xy} 在此点连续：$f'_{xy}=f'_{yx}$.

§11.2 全微分

<可微的定义> 存在与 $\Delta x, \Delta y$ 无关的常数 A.B. 使 $f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)=A\Delta x+B\Delta y+o(\sqrt{\Delta x^2+\Delta y^2})$

偏导数连续 ⇒ 可微 ⇒ 连续
　　　　　　　　⇒ 偏导数存在.
　　　　　　　　⇒ 任何方向的方向导数存在

复合函数：$\begin{cases}\text{一阶微分有形式不变性} \\ \text{高　　　没有}\end{cases}$

$u=f(x,y)$　　$du=f'_x dx+f'_y dy$　　$d^n u=\sum_{k=0}^{n} C_n^k \frac{\partial^n u}{\partial x^k \partial y^{n-k}}\Delta x^k \Delta y^{n-k}$

$$d^2 f(x_1,\cdots,x_n)=(dx_1,\cdots,dx_n)\begin{pmatrix}\frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 u}{\partial x_n\partial x_1} & \cdots & \frac{\partial^2 u}{\partial x_n^2}\end{pmatrix}\begin{pmatrix}dx_1 \\ \vdots \\ dx_n\end{pmatrix}$$

Hesse 矩阵.

$$\frac{\partial(f_1,\cdots,f_m)}{\partial(x_1,\cdots,x_n)}\triangleq\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}_{m\times n}$$

Jacobi 矩阵.

§11.3 方向导数与梯度

<方向导数> $\frac{\partial f}{\partial \vec{l}}(X_0)=\lim\limits_{\substack{X\to X_0 \\ \text{沿} l}}\frac{f(X)-f(X_0)}{|X-X_0|}$　　l 是过 X_0 的一条射线，$\vec{l}$ 代表其方向.　　可微 ⇒ 任何方向的方向导数存在

$\vec{l}=(\cos\alpha_1,\cdots,\cos\alpha_n)$　　其中 α_i 代表 $\vec{l}$ 在 x 轴的投影　　$\cos^2\alpha_1+\cdots+\cos^2\alpha_n=1$

$\frac{\partial f}{\partial \vec{l}}(X_0)=f'_{x_1}(X_0)\cdot\cos\alpha_1+f'_{x_2}(X_0)\cdot\cos\alpha_2+\cdots+f'_{x_n}(X_0)\cdot\cos\alpha_n=\langle\nabla f(X_0),\vec{l}\rangle$　　(在 $|\vec{l}|=1$ 且 f 可微时)

<梯度> $\nabla f(x_1,\cdots,x_n)=(f'_{x_1},f'_{x_2},\cdots,f'_{x_n})$　　梯度运算法则：与导数形式一致

梯度：$\begin{cases}\text{使 } \frac{\partial u}{\partial l} \text{ 达到 max 的方向} \\ \text{使函数变化最快的方向.}\end{cases}$

$f(u_1,u_2,\cdots,u_m)\xrightarrow{u_i=u_i(x_1,\cdots,x_n)}g(x_1,x_2,\cdots,x_n)$

$(g'_{x_1},g'_{x_2},\cdots,g'_{x_n})=(f'_{u_1},f'_{u_2},\cdots,f'_{u_m})\cdot\frac{\partial(u_1,u_2,\cdots,u_m)}{\partial(x_1,x_2,\cdots,x_n)}$

§11.4 多元泰勒公式

$$f(X_0+\Delta X)=f(X_0)+df(X_0)+\frac{1}{2!}d^2f(X_0)+\cdots+\frac{1}{m!}d^mf(X_0)+\begin{cases}o(|\Delta X|^m) & |\Delta X|\to 0 \\ \frac{1}{(m+1)!}d^{m+1}f(C) & C=X_0+\theta(X-X_0)\quad 0<\theta<1\end{cases}$$

特别地，$m=2$ 时　$f(X)=f(X_0)+\langle\nabla f(X_0),\Delta X\rangle+\underline{\frac{1}{2}\Delta X\cdot H_f(X_0)\cdot\Delta X^T+o(|\Delta X|^2)}$　或：$\frac{1}{2}\Delta X\cdot H_f(X_0+\theta\Delta X)\cdot\Delta X^T$

§11.5 隐函数存在定理

▷ 方程 $f(x,y)=0$

$$\left.\begin{array}{l} f(x_0,y_0)=0 \qquad f_y'(x_0,y_0)\neq 0 \\ f(x,y) 与 f_y'(x,y) 在 (x_0,y_0) 某邻域连续 \end{array}\right\} \Rightarrow 存在唯一 \begin{array}{l}定义于 B_\delta(x_0),\\ 取值于 B_\eta(y_0)\end{array} 的函数 y(x),\ 满足: \begin{cases} y_0=y(x_0) \\ f(x,y(x))\equiv 0 \end{cases}$$

$$\left.+\ f_x'(x,y) 在 (x_0,y_0) 某邻域连续\right\} \Rightarrow y'(x) = -\frac{f_x'(x,y(x))}{f_y'(x,y(x))}$$

▷ 方程组 $\begin{cases} f_1(x_1,\cdots,x_n,y_1,\cdots,y_m)=0 \\ \cdots\cdots \\ f_m(x_1,\cdots,x_n,y_1,\cdots,y_m)=0 \end{cases}$ $\quad (X_0,Y_0)=(x_{01},\cdots,x_{n1},y_{01},\cdots,y_{0m})$ $\quad F(X,Y)=\begin{pmatrix} f_1(X,Y) \\ \vdots \\ f_m(X,Y)\end{pmatrix}$

$$\left.\begin{array}{l} (X_0,Y_0) 满足方程组. \quad \dfrac{D(f_1,\cdots,f_m)}{D(y_1,\cdots,y_m)}(X_0,Y_0)\neq 0 \\ 映射 F 与 F_{y_1}', F_{y_2}', \cdots, F_{y_m}' 均在 (X_0,Y_0) 某邻域连续 \end{array}\right\} \Rightarrow \begin{array}{c}存在唯一映射 Y=Y(X),\ 满足: \\ y_i=y_i(x_1,\cdots,x_n)\end{array} \begin{cases} Y_0=Y(X_0) \\ Y(X) 在 X_0 某邻域内连续 \end{cases}$$

$$\left.+\ F_{x_1}', F_{x_2}', \cdots, F_{x_n}' 均在 (X_0,Y_0) 某邻域连续\right\} \Rightarrow \frac{\partial(y_1,y_2,\cdots,y_m)}{\partial(x_1,x_2,\cdots,x_n)} = -\left(\frac{\partial(f_1,f_2,\cdots,f_m)}{\partial(y_1,y_2,\cdots,y_m)}\right)^{-1}\cdot\frac{\partial(f_1,f_2,\cdots,f_m)}{\partial(x_1,x_2,\cdots,x_n)}$$

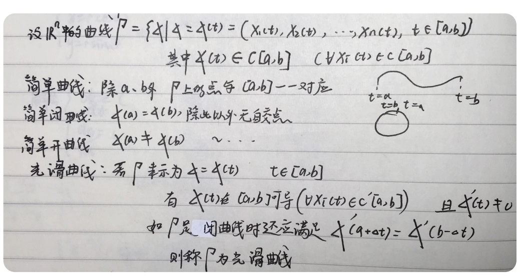
▷ 方程组 $\begin{cases} y_1=f_1(x_1,\cdots,x_n) \\ \vdots \\ y_n=f_n(x_1,\cdots,x_n) \end{cases}$ $\quad X=(x_1,\cdots,x_n)$, $Y=(y_1,\cdots,y_n)$ $\quad Y=F(X)=\begin{pmatrix} f_1 \\ \vdots \\ f_n\end{pmatrix}$ $\quad (X_0,Y_0)=(x_{01},\cdots,x_{n1},y_{01},\cdots,y_{0m})$

$$\left.\begin{array}{l} (X_0,Y_0) 满足方程组. \quad \dfrac{D(f_1,\cdots,f_n)}{D(x_1,\cdots,x_n)}(X_0)\neq 0 \\ F 与 F_{x_1}', F_{x_2}', \cdots, F_{x_n}' 均在 X_0 某邻域连续 \end{array}\right\} \Rightarrow \begin{array}{c}存在唯一映射 X=G(Y),\ 满足: \\ x_i=g_i(y_1,\cdots,y_n)\end{array} \begin{cases} V=F(U) \quad U=F(V) \\ F(g_1,\cdots,g_n)=Y \quad G(f_1,\cdots,f_n)=X \end{cases}$$

Y_0 邻域 $\quad X_0$ 邻域

$$\left.+\ G_{y_1}', G_{y_2}', \cdots, G_{y_n}' 均在 Y_0 某邻域连续\right\} \Rightarrow \frac{\partial(g_1,g_2,\cdots,g_n)}{\partial(y_1,y_2,\cdots,y_n)}(Y) = \left(\frac{\partial(f_1,f_2,\cdots,f_n)}{\partial(x_1,x_2,\cdots,x_n)}\right)^{-1}(X)$$

§11.6 曲线切线与曲面切平面


设 $\mathbb{R}^n$ 中的曲线 $\Gamma=\{x \mid x=x(t)=(x_1(t),x_2(t),\cdots,x_n(t)),\ t\in[a,b]\}$

其中 $x(t)\in C[a,b]$ $\quad (\forall x_i(t)\in C[a,b])$

简单曲线：除 a、b 外 Γ 上的点与 $[a,b]$ 一一对应

简单闭曲线：$x(a)=x(b)$，除此以外无自交点

简单开曲线 $\quad x(a)\neq x(b) \quad \sim \cdots$

光滑曲线：若 Γ 表示为 $x=x(t) \quad t\in[a,b]$

有 $x(t)$ 在 $[a,b]$ 可导 $(\forall x_i(t)\in C'[a,b])$ 且 $x'(t)\neq 0$

如 Γ 是闭曲线时还应满足 $x'(a+0t)=x'(b-0t)$

则称 Γ 为光滑曲线

Γ: $\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$ $\quad$ 过 $(x_0,y_0,z_0)=(x(t_0),y(t_0),z(t_0))$ 的

切线：$\dfrac{x-x_0}{x'(t_0)}=\dfrac{y-y_0}{y'(t_0)}=\dfrac{z-z_0}{z'(t_0)}$

法平面：$x'(t_0)\cdot(x-x_0)+y'(t_0)\cdot(y-y_0)+z'(t_0)\cdot(z-z_0)=0$

Γ: $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$ $\quad$ 切向 $\vec{\tau}=\left(\dfrac{\partial(F,G)}{\partial(y,z)},\ \dfrac{\partial(F,G)}{\partial(z,x)},\ \dfrac{\partial(F,G)}{\partial(x,y)}\right)$

S: $F(x,y,z)=0$ $\quad$ 法向量 $\vec{n}=(F_x,F_y,F_z)$ $\quad$ 切平面 $F_x\cdot(x-x_0)+F_y\cdot(y-y_0)+F_z\cdot(z-z_0)=0$

法线 $\dfrac{x-x_0}{F_x}=\dfrac{y-y_0}{F_y}=\dfrac{z-z_0}{F_z}$

S: $\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$ $\quad \vec{n}=\left(\dfrac{\partial(y,z)}{\partial(u,v)},\ \dfrac{\partial(z,x)}{\partial(u,v)},\ \dfrac{\partial(x,y)}{\partial(u,v)}\right)$

§11.7 极值理论

无条件极值:

$$\begin{cases} x_0 \text{是极小值} \Rightarrow \nabla f(x_0)=0 \ \text{且} \ H_f(x_0)\geq 0 \\ x_0 \text{是极大值} \Rightarrow \nabla f(x_0)=0 \ \text{且} \ H_f(x_0)\leq 0 \end{cases}$$

$$\begin{cases} \nabla f(x_0)=0 \ \text{且} \ H_f(x_0)>0 \Rightarrow x_0 \text{是极小值} \\ \nabla f(x_0)=0 \ \text{且} \ H_f(x_0)<0 \Rightarrow x_0 \text{是极大值} \\ \nabla f(x_0)=0 \ \text{且} \ H_f(x_0) \text{不定} \Rightarrow x_0 \text{不是极值} \end{cases}$$

条件极值: 条件为 $\varphi_i(x)=0$ 记 $L=f+\sum \lambda_i \varphi_i$

$$\begin{cases} x_0 \text{是极小值} \Rightarrow \nabla L(x_0)=0 \ \text{且} \ H_L(x_0)\geq 0 \\ x_0 \text{是极大值} \Rightarrow \nabla L(x_0)=0 \ \text{且} \ H_L(x_0)\leq 0 \end{cases}$$

$$\begin{cases} \nabla L(x_0)=0 \ \text{且} \ H_L(x_0)>0 \Rightarrow x_0 \text{是极小值} \\ \nabla L(x_0)=0 \ \text{且} \ H_L(x_0)<0 \Rightarrow x_0 \text{是极大值} \\ \nabla L(x_0)=0 \ \text{且} \ H_L(x_0) \text{不定} \Rightarrow x_0 \text{不是极值} \end{cases}$$

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练习11.1

1、(1) $z'_x = \dfrac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2} - \dfrac{1}{y}\cdot 2x\cdot \sin x^2 = -\dfrac{y}{x^2+y^2} - \dfrac{2x\cdot\sin x^2}{y}$

$z'_y = \dfrac{\frac{1}{x}}{1+(\frac{y}{x})^2} - \dfrac{\cos x^2}{y^2} = \dfrac{x}{x^2+y^2} - \dfrac{\cos x^2}{y^2}$

(2) $z = e^{x\ln x - x\ln y} + e^{\frac{y}{x}\ln x}$ $\quad z'_x = (\ln x + 1 - \ln y)e^{x\ln\frac{x}{y}} + y(\frac{1}{x^2} - \frac{\ln x}{x^2})e^{\frac{y}{x}\ln x}$

$= (\ln\frac{x}{y}+1)\cdot(\frac{x}{y})^x + \frac{y}{x^2}(1-\ln x)\cdot x^{\frac{y}{x}}$

$z'_y = -\frac{x}{y}e^{x\ln\frac{x}{y}} + \frac{\ln x}{x}\cdot e^{\frac{y}{x}\ln x} = -(\frac{x}{y})^{x+1} + x^{\frac{y}{x}-1}\cdot\ln x$

(3) $u'_x = \dfrac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $\quad u'_y = \dfrac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $\quad u'_z = \dfrac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$

(4) $u'_x = \dfrac{1}{\sqrt{1-\frac{z^2}{x^2+y^2}}}\cdot z\cdot\dfrac{(-x)}{(x^2+y^2)^{\frac{3}{2}}} = \dfrac{-xz}{(x^2+y^2)\sqrt{x^2+y^2-z^2}}$ $\quad u'_y = \dfrac{-yz}{(x^2+y^2)\sqrt{x^2+y^2-z^2}}$

$u'_z = \dfrac{1}{\sqrt{1-\frac{z^2}{x^2+y^2}}}\cdot\dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{\sqrt{x^2+y^2-z^2}}$

(5) $u'_x = -2x\cdot e^{-(x^2+y^2+z^2)}\sin(ax+by+cz) + ae^{-(x^2+y^2+z^2)}\cos(ax+by+cz)$

$u'_y = -2y\, e^{-(x^2+y^2+z^2)}\sin(ax+by+cz) + be^{-(x^2+y^2+z^2)}\cos(ax+by+cz)$

$u'_z = -2z\, e^{-(x^2+y^2+z^2)}\sin(ax+by+cz) + ce^{-(x^2+y^2+z^2)}\cos(ax+by+cz)$

(6) $u = e^{\ln x^y y^z z^x} = e^{y\ln x + z\ln y + x\ln z}$

$u'_x = x^y y^z z^x(\frac{y}{x} + \ln z)$ $\quad u'_y = x^y y^z z^x(\frac{z}{y} + \ln x)$ $\quad u'_z = x^y y^z z^x(\frac{x}{z} + \ln y)$

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2. · $x^2+y^2\neq 0$时 $\exists B_\delta(x_0,y_0)\cap(0,0)=\varphi$ 使 $\forall (x,y)\in B_\delta(x_0,y_0)$, $f(x,y)=\frac{xy}{x^2+y^2}$

$$\therefore f_x'(x,y)=\frac{y(x^2+y^2)-xy\cdot 2x}{(x^2+y^2)^2}=\frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

· $x^2+y^2=0$时. $f_x'(x,y)=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\frac{\Delta x-0}{\Delta x^2+0}-0}{\Delta x}=0$

综上: $f_x'(x,y)=\begin{cases}\frac{y(y^2-x^2)}{(x^2+y^2)^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$ 同理 $f_y'(x,y)=\begin{cases}\frac{x(x^2-y^2)}{(x^2+y^2)^2} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$

3. $\frac{\partial u}{\partial x}=\frac{1}{2\sqrt{x}}\cdot\frac{1}{\sqrt{x}+\sqrt{y}}$, $\frac{\partial u}{\partial y}=\frac{1}{2\sqrt{y}}\cdot\frac{1}{\sqrt{x}+\sqrt{y}}$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}(\sqrt{x}+\sqrt{y})\cdot\frac{1}{\sqrt{x}+\sqrt{y}}=\frac{1}{2}$$

4. (1) $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial t}=(4x+z^2)\cdot 2\cos t-z\cdot(2t-1)-(2xz-y)\cdot 3e^{-t}$

$=2(8\sin t+9e^{-2t})\cos t-3(2t-1)e^{-t}-3(12\sin t e^{-t}-t^2+t-1)e^{-t}$

(2) $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=(3x^2-y)\cos\theta+(3y^2-x)\sin\theta=3r^2(\cos^3\theta+\sin^3\theta)-2r\sin\theta\cos\theta$

$\frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}=(3x^2-y)\cdot(-r\sin\theta)+(3y^2-x)r\cos\theta=3r^3\sin\theta\cos\theta(\sin\theta-\cos\theta)-r^2\cos 2\theta$

(3) $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r}=\left(\frac{x}{\sqrt{x^2+y^2+z^2}}+\frac{1}{1+\frac{x^2+y^2}{z^2}}\cdot\frac{x}{z\sqrt{x^2+y^2}}\right)\sin\varphi\cos\theta+\left(\frac{y}{\sqrt{x^2+y^2}}+\frac{1}{1+\frac{x^2+y^2}{z^2}}\cdot\frac{y}{z\sqrt{x^2+y^2}}\right)\sin\varphi\sin\theta$

$+\left(\frac{z}{\sqrt{x^2+y^2+z^2}}+\frac{1}{1+\frac{x^2+y^2}{z^2}}\cdot\frac{(-\sqrt{x^2+y^2})}{z^2}\right)\cos\varphi=1.$ $\therefore\frac{\partial u}{\partial\varphi}=1$

$\frac{\partial u}{\partial\theta}=-\left(\sin\varphi\cos\theta+\frac{\cos\varphi\cos\theta}{r}\right)r\sin\varphi\sin\theta+\left(\sin\varphi\sin\theta+\frac{\cos\varphi\sin\theta}{r}\right)r\sin\varphi\cos\theta=0$

5. $z_x' = y + f(\frac{y}{x}) + x\cdot(\frac{-y}{x^2})\cdot f'(\frac{y}{x}) = y + f(\frac{y}{x}) - \frac{y}{x} f'(\frac{y}{x})$

$z_y' = x + x\cdot\frac{1}{x}\cdot f'(\frac{y}{x}) = x + f'(\frac{y}{x})$

$\therefore x\cdot z_x' + y\cdot z_y' = xy + xf(\frac{y}{x}) - yf'(\frac{y}{x}) + xy + y\cdot f'(\frac{y}{x}) = 2xy + xf(\frac{y}{x}) = xy + z$.

6. 令 $s = \frac{y-x}{xy}$, $t = \frac{z-x}{xz}$.

$\frac{\partial u}{\partial x} = -\frac{1}{x^2}\cdot f_s' - \frac{1}{x^2}\cdot f_t'$ $\quad \frac{\partial u}{\partial y} = \frac{1}{y^2} f_s'$ $\quad \frac{\partial u}{\partial z} = \frac{1}{z^2} f_t'$

$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = -f_s' - f_t' + f_s' + f_t' = 0$

7. (1) $\frac{\partial u}{\partial x} = \frac{a}{ax+by+cz}$ $\quad \frac{\partial^2 u}{\partial x^2} = \frac{-a^2}{(ax+by+cz)^2}$ $\quad \frac{\partial^3 u}{\partial x^3} = \frac{2a^3}{(ax+by+cz)^3}$ $\quad \frac{\partial^4 u}{\partial x^4} = \frac{-6a^4}{(ax+by+cz)^4}$

$\frac{\partial u}{\partial y} = \frac{b}{ax+by+cz}$ $\quad \frac{\partial^2 u}{\partial y^2} = \frac{-b^2}{(ax+by+cz)^2}$ $\quad \frac{\partial^3 u}{\partial x\partial y^2} = \frac{2ab^2}{(ax+by+cz)^3}$ $\quad \frac{\partial^4 u}{\partial x^2\partial y^2} = \frac{-6a^2b^2}{(ax+by+cz)^4}$

(2) $\frac{\partial^r u}{\partial z^r} = xy(z+r)e^{x+y+z}$. $\quad (u_z' = xy(z+1)e^{x+y+z},\ u_z'' = xy(z+2)e^{x+y+z},\ \cdots\cdots)$

$\frac{\partial^{q+r} u}{\partial y^q\partial z^r} = x(y+q)(z+r)e^{x+y+z}$

$\frac{\partial^{p+q+r} u}{\partial x^p\partial y^q\partial z^r} = (x+p)(y+q)(z+r)e^{x+y+z}$

8. $\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x_i} = \frac{x_i}{r} \cdot f'(r)$

$\frac{\partial^2 u}{\partial x_i^2} = \frac{r - \frac{x_i^2}{r}}{r^2} f'(r) + \frac{x_i^2}{r^2} f''(r) = \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) f'(r) + \frac{x_i^2}{r^2} f''(r)$

$\therefore \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{n}{r} f'(r) - \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} f''(r) = f''(r) + \frac{n-1}{r} f'(r)$

9. $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} = e^s \frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial s^2} = e^s \frac{\partial u}{\partial x} + x e^s \frac{\partial^2 u}{\partial x^2} = x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} = e^t \frac{\partial u}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial t^2} = e^t \frac{\partial u}{\partial y} + y e^t \frac{\partial^2 u}{\partial t^2} = y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$

$\therefore x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$

练习11.2

1. (1) $\frac{\partial u}{\partial x}=yz\cos xz \quad \frac{\partial u}{\partial y}=\sin xz \quad \frac{\partial u}{\partial z}=xy\cos xz \Rightarrow du=yz\cos(xz)dx+\sin(xz)dy + xy\cos(xz)dz$

(2) $\frac{\partial u}{\partial x}=e^y \quad \frac{\partial u}{\partial y}=xe^y \Rightarrow du=e^y dx+xe^y dy$

(3) $\frac{\partial u}{\partial x}=x^y y^z z^x(\frac{y}{x}+\ln z)$; $\frac{\partial u}{\partial y}=x^y y^z z^x(\frac{z}{y}+\ln x)$; $\frac{\partial u}{\partial z}=x^y y^z z^x(\frac{x}{z}+\ln y)$

$\Rightarrow du=x^y y^z z^x\left[(\frac{y}{x}+\ln z)dx+(\frac{z}{y}+\ln x)dy+(\frac{x}{z}+\ln y)dz\right]$

(4) $\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2+z^2+\sqrt{x^2+y^2+z^2}} \quad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2+z^2+\sqrt{x^2+y^2+z^2}} \quad \frac{\partial u}{\partial z}=\frac{z}{x^2+y^2+z^2+\sqrt{x^2+y^2+z^2}}$

$\Rightarrow du=\frac{1}{x^2+y^2+z^2+\sqrt{x^2+y^2+z^2}}(xdx+ydy+zdz)$

2. (1) $du=\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}dx+\frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}dy=f_t'\cdot\frac{x}{\sqrt{x^2+y^2}}dx+f_t'\cdot\frac{y}{\sqrt{x^2+y^2}}dy=f_t'\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot(xdx+ydy)$

(2) $dw=\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy+\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}\right)dz$

$=(f_u'+2xf_v')dx+(f_u'+2y\cdot f_v')dy+(f_u'+2z\cdot f_v')dz.$

3. (1) $\frac{\partial(f_1,f_2)}{\partial(x,y)}=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

(2) $\frac{\partial(f_1,f_2,f_3)}{\partial(u,v)}=\begin{pmatrix}\cos v & -u\sin v\\ \sin v & u\cos v\\ 0 & 1\end{pmatrix}$ $(u,v)=(1,\pi)$时，矩阵为$\begin{pmatrix}-1 & 0\\ 0 & -1\\ 0 & 1\end{pmatrix}$

练习11.3

1. $\frac{\partial f}{\partial \vec{l}}(0,0)=\lim\limits_{t\to 0^+}\frac{f(t\cos\theta,t\sin\theta)-f(0,0)}{t}=\lim\limits_{t\to 0^+}\frac{(t\cos\theta)^3+(t\sin\theta)^3}{t(t^2\cos^2\theta+t^2\sin^2\theta)}=\cos^3\theta+\sin^3\theta$

注：f在0处不可微，不可应用 $\langle \nabla f, \vec{l}\rangle$.

2. $\nabla f=(2x-y,2y-x)\big|_{(1,1)}=(1,1)$　　$\vec{l}=(\cos\theta,\sin\theta)$　　$\therefore \frac{\partial f}{\partial \vec{l}}(1,1)=\sin\theta+\cos\theta=\sqrt{2}\sin(\theta+\frac{\pi}{4})$

(1) $\theta=\frac{\pi}{4}$时 $\frac{\partial f}{\partial \vec{l}}(1,1)$最大. $\vec{l}=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$

(2) $\theta=\frac{5}{4}\pi$时 $\frac{\partial f}{\partial \vec{l}}(1,1)$最小. $\vec{l}=(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$

(3) $\theta=\frac{3}{4}\pi$或$\frac{7}{4}\pi$时 $\frac{\partial f}{\partial \vec{l}}(1,1)$为0. $\vec{l}=(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$或$(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$

3. (1) $\nabla u=(yz,xz,xy)\big|_{(1,1,1)}=(1,1,1)$　　范数为$\sqrt{1^2+1^2+1^2}=3$

(2) $\nabla u=(2x-y,2y-x,6z+6)\big|_{(1,2,3)}=(0,3,24)$　　范数为$\sqrt{3^2+24^2}=3\sqrt{65}$

4. $\nabla f=(ay^2+3cx^2z^2,\ 2axy+bz,\ by+2czx^3)\big|_{(1,2,-1)}=(4a+3c,4a-b,2b-2c)$

$\vec{l}=(0,0,1)$　　$\therefore \frac{\partial f}{\partial \vec{l}}=\langle \nabla f,\vec{l}\rangle=2b-2c$

$\because \nabla f$是方向导数的最大方向　$\therefore 4a+3c=0$且$4a-b=0$时，$\frac{\partial f}{\partial \vec{l}}$取到最大值64.

由$\begin{cases}4a+3c=0\\4a-b=0\\2b-2c=64\end{cases}\Rightarrow\begin{cases}a=6\\b=24\\c=-8\end{cases}$

5. $\frac{1}{r}=\frac{1}{\sqrt{x^2+y^2+z^2}}$ $\quad \nabla\frac{1}{r}=\left(\frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}},\frac{-y}{(x^2+y^2+z^2)^{\frac{3}{2}}},\frac{-z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)=-\frac{1}{r^3}(x,y,z)=-\frac{1}{r^3}\cdot\vec{r}$

6. $\nabla u=-\left(\frac{x-a}{(x-a)^2+(y-b)^2+(z-c)^2},\frac{y-b}{(x-a)^2+(y-b)^2+(z-c)^2},\frac{z-c}{(x-a)^2+(y-b)^2+(z-c)^2}\right)$

$|\nabla u|=\frac{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}{(x-a)^2+(y-b)^2+(z-c)^2}=\frac{1}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}$ 当$|\Delta u|=1$时，$(x-a)^2+(y-b)^2+(z-c)^2=1$.

即以(a,b,c)为球心，半径为1的球面上的点能使$|\Delta u|=1$.

7. $\begin{cases}x=r\sin\varphi\cos\theta\\ y=r\sin\varphi\sin\theta\\ z=r\cos\varphi\end{cases}\Rightarrow\begin{cases}r=\sqrt{x^2+y^2+z^2}\\ \varphi=\arccos\frac{z}{\sqrt{x^2+y^2+z^2}}\\ \theta=\arctan\frac{y}{x}\end{cases}$

$\therefore\nabla r=\frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,z)$ $\quad\therefore|\nabla r|=\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot\sqrt{x^2+y^2+z^2}=1$

$\therefore\nabla\varphi=-\frac{1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}}\cdot\left(\frac{-xz}{(x^2+y^2+z^2)^{\frac{3}{2}}},\frac{-yz}{(x^2+y^2+z^2)^{\frac{3}{2}}},\frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)=\frac{-1}{r^2\cdot\sqrt{x^2+y^2}}(-xz,-yz,x^2+y^2)$

$\therefore|\nabla\varphi|=\frac{1}{r^2\sqrt{x^2+y^2}}\cdot\sqrt{x^2z^2+y^2z^2+(x^2+y^2)^2}=\frac{1}{r}=\frac{1}{r_0}$

$\therefore\nabla\theta=\left(-\frac{y}{x^2}\cdot\frac{1}{1+\frac{y^2}{x^2}},\frac{1}{x}\cdot\frac{1}{1+\frac{y^2}{x^2}},0\right)=\left(-\frac{y}{x^2+y^2},\frac{x}{x^2+y^2},0\right)$

$\therefore|\nabla\theta|=\frac{1}{x^2+y^2}\sqrt{x^2+y^2}=\frac{1}{\sqrt{x^2+y^2}}=\frac{1}{r_0\sin\varphi_0}$

练习11.4

1. (1) $\nabla f=(-\sin x\cos y,-\cos x\sin y)$ $H_f=\begin{pmatrix}-\cos x\cos y & \sin x\sin y\\ \sin x\sin y & -\cos x\cos y\end{pmatrix}$

$\therefore f(x,y)=f(0,0)+\langle\nabla f,\Delta x\rangle+\frac{1}{2}\Delta x\cdot H_f\cdot\Delta x^T+o(\sqrt{x^2+y^2})$

$=1+0+\frac{1}{2}(x,y)\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=1-\frac{1}{2}x^2-\frac{1}{2}y^2+o(\sqrt{x^2+y^2})$

(2) $\nabla f=(-\frac{\sin x}{\cos y},\frac{\cos x\sin y}{\cos^2 y})$ $H_f=\begin{pmatrix}-\frac{\cos x}{\cos y} & \frac{\sin x\sin y}{\cos^2 y}\\ \frac{\sin x\sin y}{\cos^2 y} & \frac{\cos x(1+\sin^2 y)}{\cos^3 y}\end{pmatrix}$

$\therefore f(x,y)=1+0+\frac{1}{2}(x,y)\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=1-\frac{1}{2}x^2+\frac{1}{2}y^2+o(\sqrt{x^2+y^2})$

(3) $\nabla f=(4x-y-6,2y-x-3)$ $H_f=\begin{pmatrix}4&-1\\-1&2\end{pmatrix}$

$\therefore f(x,y)=f(1,-2)+\langle(0,-8),((x-1),(x+2))\rangle+\frac{1}{2}(x-1,x+2)\begin{pmatrix}4&-1\\-1&2\end{pmatrix}\begin{pmatrix}x-1\\x+2\end{pmatrix}$

$=13-8(x+2)+\frac{1}{2}\left[4(x-1)^2+2(x+2)^2-2(x-1)(x+2)\right]+o\left(\sqrt{(x-1)^2+(x+2)^2}\right)$

2. $\because\frac{\partial^k}{\partial y^k}f(x,y)=\frac{(-1)^{k-1}\cdot(k-1)e^x}{(y+1)^k}$ $\therefore\frac{\partial^n f}{\partial x^k\partial y^{n-k}}=\frac{(-1)^{n-k-1}\cdot(n-k-1)e^x}{(y+1)^{n-k-1}}$

$\therefore d^n f=\sum_{k=0}^{n}C_n^k\cdot\frac{\partial^n f}{\partial x^k\partial y^{n-k}}\cdot x^k\cdot y^{n-k}=\sum_{k=0}^{n}(-1)^{n-k-1}\cdot(n-k-1)\cdot C_n^k\cdot\frac{x^k\cdot y^{n-k}\cdot e^x}{(y+1)^{n-k-1}}$

$\therefore d^n f(0,0)=\sum_{k=0}^{n}(-1)^{n-k-1}(n-k-1)C_n^k\cdot x^k\cdot y^{n-k}=\sum_{k=0}^{n}(-1)^{k-1}\cdot(k-1)C_n^k\cdot x^{n-k}y^k$

$\therefore f(x,y)=\sum_{n=1}^{m}\sum_{k=0}^{n}\frac{(-1)^{k-1}\cdot(k-1)C_n^k}{n!}\cdot x^{n-k}y^k+o(\sqrt{(x^2+y^2)^m})$

练习11.5

1. (1) 记 $F(x,y,z)=x+y+z-e^{-(x+y+z)}$

$F_x=F_y=F_z=1+e^{-(x+y+z)}$

$\therefore Z_x=-\frac{F_x}{F_z}=-1$, $Z_y=-\frac{F_y}{F_z}=-1$ $\quad Z_{xx}=Z_{xy}=Z_{yy}=0$

(2) 记 $F(x,y,z)=\sin(x+y)+\sin(y+z)-1$.

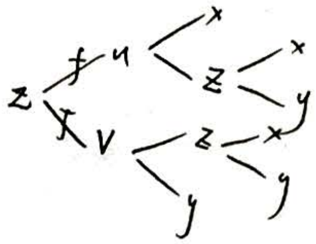
$F_x=\cos(x+y)$ $\quad F_y=\cos(x+y)+\cos(y+z)$ $\quad F_z=\cos(y+z)$

$\therefore Z_x=-\frac{F_x}{F_z}=-\frac{\cos(x+y)}{\cos(y+z)}$ $\quad Z_y=-\frac{F_y}{F_z}=-\frac{\cos(x+y)}{\cos(y+z)}-1$.

$Z_{xx}=\frac{\sin(x+y)}{\cos(y+z)}$ $\quad Z_{xy}=\frac{\sin(x+y)\cos(y+z)-\sin(y+z)\cos(x+y)}{\cos^2(y+z)}$

$Z_{yy}=\frac{\sin(x+y)\cos(y+z)-\sin(y+z)\cos(x+y)}{\cos^2(y+z)}-1$

练习11.5.



2、(1) $dz = z_x dx + z_y dy$ 记 $u = xz$ $v = z - y$

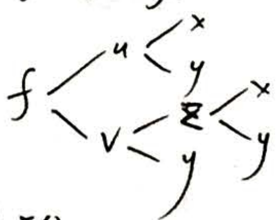
∴ $z_x = f_u\cdot u_x + f_u\cdot u_z\cdot z_x + f_v\cdot v_z\cdot z_x = f_u\cdot z + f_u\cdot x\cdot z_x + f_v\cdot z_x = f_u(z + x z_x) + f_v\cdot z_x$

∴ $(1 - f_u x - f_v) z_x = z f_u$ ∴ $z_x = \frac{z f_u}{1 - x f_u - f_v}$ 同理 $z_y = \frac{f_v}{x f_u + f_v - 1}$

∴ $dz = \frac{-z f_u\cdot dx + f_v\cdot dy}{x f_u + f_v - 1}$

(2) ∵ $f(xy, yz) = 0$ 令 $u = xy = u(x,y)$ $v = yz = v\,z(x,y) = v(x,y)$

两边对x求导. $f(u,v) = 0$



$f_u\cdot u_x + f_v\cdot v_z\cdot z_x = 0$ ∴ $f_u\cdot y + f_v\cdot y\cdot z_x = 0$

∴ $f_u + f_v\cdot z_x = 0$ 即 $f_u(u,v) + f_v(u,v)\cdot z_x = 0$

两边对x求导: $f_{uu}\cdot u_x + f_{uv}\cdot v_z\cdot z_x + (f_{vu}\cdot u_x + f_{vv}\cdot v_z\cdot z_x) z_x + f_v\cdot z_{xx} = 0$

∴ $f_{uu}\cdot y + f_{uv}\cdot y\cdot z_x + (f_{uv}\cdot y + f_{vv}\cdot y\cdot z_x) z_x + f_v\cdot z_{xx} = 0$

∴ $\frac{\partial^2 z}{\partial x^2} = z_{xx} = \frac{f_{uu}\cdot y + (f_{uv}\cdot y + f_{uv}\cdot y) z_x + f_{vv}\cdot y\cdot (z_x)^2}{-f_v}$

$= \frac{f_{uu}\cdot y + 2 f_{uv}\cdot y\cdot \frac{-f_u}{f_v} + f_{vv}\cdot y\cdot \left(\frac{f_u}{f_v}\right)^2}{-f_v}$

$= -\frac{y\cdot f_{uu}\cdot (f_v)^3 - 2y f_u f_v f_{uv} + y\cdot f_{vv}(f_u)^2}{(f_v)^3}$

3. $\frac{\partial x}{\partial y}=-\frac{f_y}{f_x}$ $\frac{\partial y}{\partial z}=-\frac{f_z}{f_y}$ $\frac{\partial z}{\partial x}=-\frac{f_x}{f_z}$

$\therefore \frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial x}=-\frac{\cancel{f_y}}{\cancel{f_x}}\cdot\frac{\cancel{f_z}}{\cancel{f_y}}\cdot\frac{\cancel{f_x}}{\cancel{f_z}}=-1$

4. 记$F(x,y,z)=z^2+xy-a$，$G(x,y,z)=z^2+x^2-y^2-b$

$\frac{\partial F}{\partial x}=y$ $\frac{\partial F}{\partial y}=x$ $\frac{\partial F}{\partial z}=2z$

$\frac{\partial G}{\partial x}=2x$ $\frac{\partial G}{\partial y}=-2y$ $\frac{\partial G}{\partial z}=2z$

(1) 令 $\frac{D(F,G)}{D(x,y)}\neq 0$ 即 $\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{vmatrix}\neq 0$ 即 $\begin{vmatrix} y & x\\ 2x & -2y\end{vmatrix}\neq 0$

$\therefore x^2+y^2\neq 0$ 为条件.

(2) 令 $\frac{D(F,G)}{D(x,z)}\neq 0$ 即 $\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial z}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z}\end{vmatrix}\neq 0$ 即 $\begin{vmatrix} y & 2z\\ 2x & 2z\end{vmatrix}\neq 0$

即 $z(y-2x)\neq 0$ 为条件.

5.

(1) 方程组

对x求偏导:
$$\begin{cases} 2u\dfrac{\partial u}{\partial x}-\dfrac{\partial v}{\partial x}=3 \\ 2u\dfrac{\partial u}{\partial x}-4v\dfrac{\partial v}{\partial x}=1 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial v}{\partial x}=\dfrac{2}{4v-1} \\ \dfrac{\partial u}{\partial x}=\dfrac{12v-1}{2u(4v-1)} \end{cases}$$

对y求偏导:
$$\begin{cases} 2u\dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial y}=1 \\ 2u\dfrac{\partial u}{\partial y}-4v\dfrac{\partial v}{\partial y}=-2 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial v}{\partial y}=\dfrac{3}{4v-1} \\ \dfrac{\partial u}{\partial y}=\dfrac{2v+1}{u(4v-1)} \end{cases}$$

(2) $\dfrac{x}{y}=\dfrac{\cos\theta}{\sin\theta}=\dfrac{\cos\theta}{z}$ $\therefore \cos\theta=\dfrac{xz}{y}$. 又 $\sin\theta=z$

$\therefore z^2+\dfrac{x^2z^2}{y^2}=1$ 即 $y^2z^2+x^2z^2=y^2$

对x求偏导: $2z\cdot y^2\dfrac{\partial z}{\partial x}+2x\cdot z^2+2z\cdot x^2\cdot\dfrac{\partial z}{\partial x}=0$

$\therefore 2z\cos^2\varphi\dfrac{\partial z}{\partial x}=-2xz^2$ $\therefore \dfrac{\partial z}{\partial x}=-\dfrac{\sin\theta\cos\theta}{\cos\varphi}$

对y求偏导: $2y\cdot z^2+2z\cdot y^2\cdot\dfrac{\partial z}{\partial y}+2z\cdot x^2\dfrac{\partial z}{\partial y}=2y$

$\therefore \cos^2\varphi\dfrac{\partial z}{\partial y}=2y-2y\cdot z^2$ $\therefore \dfrac{\partial z}{\partial y}=\dfrac{\cos^2\theta}{\cos\varphi}$

练习11.6

1. (1) $\frac{\partial x}{\partial t}=2a\sin t\cos t$ $\frac{\partial y}{\partial t}=b(\cos^2 t-\sin^2 t)$ $\frac{\partial z}{\partial t}=-2c\sin t\cos t$

$x(\frac{\pi}{4})=\frac{a}{2}$ $y(\frac{\pi}{4})=\frac{b}{2}$ $z(\frac{\pi}{4})=\frac{c}{2}$

$x'(\frac{\pi}{4})=a$ $y'(\frac{\pi}{4})=0$ $z'(\frac{\pi}{4})=-c$

∴切线: $\frac{x-\frac{a}{2}}{a}=\frac{y-\frac{b}{2}}{0}=\frac{z-\frac{c}{2}}{-c}$

切平面: $a(x-\frac{a}{2})-c(z-\frac{c}{2})=0$

(2) 记 $F(x,y,z)=3x^2y+yz+4$, $G(x,y,z)=2xz-x^2y-3$

∴ $\frac{\partial F}{\partial x}=6xy$ $\frac{\partial F}{\partial y}=3x^2+z$ $\frac{\partial F}{\partial z}=y$
$=-6$ $=4$ $=-1$

$\frac{\partial G}{\partial x}=2z-2xy$ $\frac{\partial G}{\partial y}=-x^2$ $\frac{\partial G}{\partial z}=2x$
$=4$ $=-1$ $=2$ 代入点 $(1,-1,1)$

切线 $\vec{t}=\left\{\frac{D(F,G)}{D(y,z)},\frac{D(F,G)}{D(z,x)},\frac{D(F,G)}{D(x,y)}\right\}=\left\{\begin{vmatrix}4&-1\\-1&2\end{vmatrix},\begin{vmatrix}-1&-6\\2&4\end{vmatrix},\begin{vmatrix}-6&4\\4&-1\end{vmatrix}\right\}=(7,8,-10)$

切线: $\frac{x-1}{7}=\frac{y+1}{8}=\frac{z-1}{-10}$

法平面: $7(x-1)+8(y+1)-10(z-1)=0$

(3) 记 $F(x,y,z)=x+y+z-3$ $G(x,y,z)=x^2-y^2+2z^2-2$

∴ $\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=1$ $\frac{\partial G}{\partial x}=2x$ $\frac{\partial G}{\partial y}=-2y$ $\frac{\partial G}{\partial z}=4z$

在 $(1,1,1)$ 点 $=2$ $=-2$ $=4$

切向 $\vec{t}=\left\{\frac{D(F,G)}{D(y,z)},\frac{D(F,G)}{D(z,x)},\frac{D(F,G)}{D(x,y)}\right\}=\left\{\begin{vmatrix}1&1\\-2&4\end{vmatrix},\begin{vmatrix}1&1\\4&2\end{vmatrix},\begin{vmatrix}1&1\\2&-2\end{vmatrix}\right\}=(6,-2,-4)$

切线: $\frac{x-1}{6}=\frac{y+1}{-2}=\frac{z+4}{-4}$

法平面: $6(x-1)-2(y+1)-4(z+4)=0$

2、证：$\frac{\partial x}{\partial t}=-a\sin t$ $\frac{\partial y}{\partial t}=a\cos t$ $\frac{\partial z}{\partial t}=b$ 切向 $\vec{l}=(-a\sin t, a\cos t, b)$

取z轴的向量 $\vec{O_z}=(0,0,1)$ $\cos\langle\vec{l},\vec{O_z}\rangle=\frac{\vec{l}\cdot\vec{O_z}}{|\vec{l}|\cdot|\vec{O_z}|}=\frac{b}{\sqrt{a^2+b^2}}$

∴无论t如何，$\langle l,\vec{O_z}\rangle\equiv\arccos\frac{b}{\sqrt{a^2+b^2}}$.

3、(1) 记 $F(x,y,z)=2^{\frac{x}{z}}+2^{\frac{y}{z}}-8$

∴ $F_x=2^{\frac{x}{z}}\cdot\ln 2\cdot\frac{1}{z}$ $F_y=2^{\frac{y}{x}}\cdot\ln 2\cdot\frac{1}{z}$ $F_z=2^{\frac{x}{z}}\cdot\ln 2\cdot\frac{-x}{z^2}+2^{\frac{y}{x}}\cdot\ln 2\cdot\frac{-y}{z^2}$

∴ $(F_x,F_y,F_z)\big|_{(2,2,1)}=(4\ln 2,4\ln 2,-16\ln 2)$ 记曲面法向为 $\vec{l}=(1,1,-4)$

∴切平面：$(x-2)+(y-2)-4(z-1)=0$ 法线：$\frac{x-2}{1}=\frac{y-2}{1}=\frac{z-1}{-4}$

(2) 记 $f(x,y)=\arctan\frac{y}{x}$ $f_x=\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}$ $f_y=\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}$

∴ $(-f_x,-f_y,1)\big|_{(1,1,\frac{\pi}{4})}=(\frac{1}{2},\frac{-1}{2},1)$ 记曲面法向为 $\vec{l}=(1,-1,2)$

切平面：$(x-1)-(y-1)+2(z-\frac{\pi}{4})=0$ 法线：$\frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{4}}{2}$

(3)

$x^2+y^2=r^2=\frac{z^2}{\cot^2\alpha}$ 记 $F(x,y,z)=x^2\cot^2\alpha+y^2\cot^2\alpha-z^2=0$，记 $P_0(r_0\cos\varphi_0, r_0\sin\varphi_0, r_0\cot\alpha)$

∵ $F_x=2x\cot^2\alpha$ $F_y=2y\cot^2\alpha$，$F_z=-2z$

∴ $(F_x,F_y,F_z)\big|_{P_0}=(2r_0\cos\varphi_0\cot^2\alpha, 2r_0\sin\varphi_0\cot^2\alpha, -2r_0\cot\alpha)$ 记法向 $\vec{l}=(\cos\varphi_0\cot\alpha, \sin\varphi_0\cot\alpha, -1)$

切平面：$\cos\varphi_0\cot\alpha(x-r_0\cos\varphi_0)+\sin\varphi_0\cot\alpha(y-r_0\sin\varphi_0)-(z-r_0\cot\alpha)=0$

法线：$\frac{x-r_0\cos\varphi_0}{\cos\varphi_0\cot\alpha}=\frac{y-r_0\sin\varphi}{\sin\varphi_0\cot\alpha}=\frac{z-r_0\cot\alpha}{-1}$

(4) $\frac{x}{y}=\frac{\cos\frac{z}{a}}{\sin\frac{z}{a}}$ 记 $F(x,y,z)=x\sin\frac{z}{a}-y\cos\frac{z}{a}=0$ 记 $P_0(u_0\cos v_0, u_0\sin v_0, av_0)$

∴ $F_x=\sin\frac{z}{a}$ $F_y=-\cos\frac{z}{a}$ $F_z=\frac{x}{a}\cos\frac{z}{a}+\frac{y}{a}\sin\frac{z}{a}$

记法向 $\vec{l}=(F_x,F_y,F_z)\big|_{P_0}=(\sin v_0,-\cos v_0,\frac{u_0}{a})\Rightarrow(a\sin v_0,-a\cos v_0,u_0)$

切平面：$a\sin v_0(x-u_0\cos v_0)-a\cos v_0(y-u_0\sin v_0)+u_0(z-av_0)=0$

法线：$\frac{x-u_0\cos v_0}{a\sin v_0}=\frac{y-u_0\sin v_0}{-a\cos v_0}=\frac{z-av_0}{u_0}$

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4. 记 $z=f(x,y)$ ∴ $f_x=2x$, $f_y=2y$ $\vec{n}=(-f_x,-f_y,1)=(-2x,-2y,1)$

曲面上过任一点 $P_0(x_0,y_0,z_0)$ 的法线为 $\frac{x-x_0}{-2x_0}=\frac{y-y_0}{-2y_0}=\frac{z-z_0}{1}$

① 若 $x_0^2+y_0^2\neq 0$ 代入点 $(0,0,0)$ ∴ $\frac{1}{2}=\frac{1}{2}=-z_0$ $z_0=-\frac{1}{2}$

∴ 满足 $\begin{cases} x_0^2+y_0^2=\frac{1}{2} \\ z_0=-\frac{1}{2} \end{cases}$ 的点 P_0 符合条件

② 若 $x_0=0$ 且 $y_0=0$ 法线为z轴，通过原点，符合条件。

此时 $z_0=x_0^2+y_0^2-1=-1$ ∴ $P_0(0,0,-1)$ 亦符合条件。

5. 令 $F(x,y,z)=x^2+y^2+z^2-16$ $G(x,y,z)=x^2+(y-5)^2+z^2-9$

$\vec{n_F}=(F_x,F_y,F_z)=(x,y,z)$ $\vec{n_G}=(G_x,G_y,G_z)=(x,y-5,z)$

令 $F=G$ ∴ $y^2=(y-5)^2+7$ ∴ $10y=32$ $y=\frac{16}{5}$

∴ F与G任一交点为 $P_0(x_0,\frac{16}{5},z_0)$ ∴ $F(P_0)=x_0^2+z_0^2-\frac{144}{25}=0$

$\vec{n_F}|_{P_0}\cdot\vec{n_G}|_{P_0}=(x_0,\frac{16}{5},z_0)\cdot(x_0,-\frac{9}{5},z_0)$

$=x_0^2+z_0^2-\frac{144}{25}=F(P_0)=0.$

∴ $F=0$ 与 $G=0$ 正交.

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练习11.7

1、(1)

$$\begin{cases} f_x = y(x^2+y^2-4)+2x^2y=0 \\ f_y = x(x^2+y^2-4)+2xy^2=0 \end{cases} \Rightarrow X_1(0,0)\ X_2(0,2)\ X_3(0,-2)\ X_4(-2,0)\ X_5(2,0)$$

$X_6(1,1)$ $X_7(-1,1)$ $X_8(1,-1)$ $X_9(-1,-1)$

$\because f(X_1)=f(X_2)=\cdots=f(X_5)=0$ $f(X_6)=f(X_9)=-2$ $f(X_7)=f(X_8)=2$

经检验，f 极小值为 -2，极大值为 2（检验 X_6、X_9、X_7、X_8 时 H_f 矩阵正负定性）

(2)

$$\begin{cases} f_x = \cos x - \sin x - \sin(x-y)=0 \\ f_y = \sin(x-y)=0 \end{cases} \Rightarrow X\left(\frac{\pi}{4}+k_1\pi,\ \frac{5}{4}\pi+k_2\pi\right) \quad k_1、k_2 \in Z$$

$f(X)=\sqrt{2}\sin\left(\frac{\pi}{2}+k_1\pi\right)+\cos\left[\pi+(k_2-k_1)\pi\right]$

当 k_1 为偶，k_2 为偶时，$f(X)=\sqrt{2}-1$

k_1 为偶，k_2 为奇时，$f(X)=\sqrt{2}+1$

k_1 为奇，k_2 为奇时，$f(X)=-\sqrt{2}+1$

k_1 为奇，k_2 为偶时，$f(X)=-\sqrt{2}-1$

经检验 f 极大值为 $1+\sqrt{2}$

极小值为 $-1-\sqrt{2}$

(3)

$$\begin{cases} f_x = y\cdot\ln(x^2+y^2)+\dfrac{2x^2y}{x^2+y^2}=0 \\ f_y = x\cdot\ln(x^2+y^2)+\dfrac{2xy^2}{x^2+y^2}=0 \end{cases} \Rightarrow X_1(1,0)\ X_2(-1,0)\ X_3(0,1)\ X_4(0,-1)$$

$X_5\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$ $X_6\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$ $X_7\left(\frac{-1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$ $X_8\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$

$f(X_1)=f(X_2)=f(X_3)=f(X_4)=0$

$f(X_5)=f(X_8)=\frac{1}{2e}$ $f(X_6)=f(X_7)=-\frac{1}{2e}$

经检验，f 极小值为 $-\frac{1}{2e}$，极大值为 $\frac{1}{2e}$.

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2.

(1) 令 $L(x,y,z)=x-2y+2z+\lambda(x^2+y^2+z^2-1)$

$$\begin{cases} L_x=1+2\lambda x=0 \\ L_y=-2+2\lambda y=0 \\ L_z=2+2\lambda z=0 \\ x^2+y^2+z^2=1 \end{cases} \Rightarrow \begin{array}{lll} \lambda_1=\frac{3}{2} & P_1(-\frac{1}{3},\frac{2}{3},-\frac{2}{3}) & f(P_1)=-3 \\ \lambda_2=-\frac{3}{2} & P_2(\frac{1}{3},-\frac{2}{3},\frac{2}{3}) & f(P_2)=3 \end{array}$$

$\because H_L=\begin{pmatrix} 2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 \\ 0 & 0 & 2\lambda \end{pmatrix}$ 顺序主子式分别为 2λ、$4\lambda^2$、$8\lambda^3$

$\therefore \lambda=\frac{3}{2}$时 $H_L>0$. $\therefore f(P_1)=-3$ 为极小值

$\lambda=-\frac{3}{2}$时 $H_L<0$ $\therefore f(P_2)=3$ 为极大值.

(2) 令 $L(x,y,z)=xyz+\lambda(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{a})$

令 $\begin{cases} L_x=yz-\frac{\lambda}{x^2}=0 \\ L_y=xz-\frac{\lambda}{y^2}=0 \\ L_z=xy-\frac{\lambda}{z^2}=0 \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{a} \end{cases} \Rightarrow \lambda=81a^4,\ x=y=z=3a$

$\because H_L=\begin{pmatrix} \frac{2\lambda}{x^3} & z & y \\ z & \frac{2\lambda}{y^3} & x \\ y & x & \frac{2\lambda}{z^3} \end{pmatrix}$ 顺序主子式分别为 $\frac{2\lambda}{x^3}$、$\frac{4\lambda^2}{x^3y^3}-z^2$、$\frac{2\lambda}{a}+2xyz+\frac{8\lambda^3}{x^3y^3z^3}$

代入上述结果: $6a$、$27a^2$、$540a^3$

$\because a>0$ $\therefore H_L>0$ $\therefore f(3a,3a,3a)=27a^3$ 是 f 的极小值.

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2.(3)

记 $L(x,y,z)=xyz+\lambda_1(x^2+y^2+z^2-1)+\lambda_2(x+y+z)$

令
$$\begin{cases} L_x=yz+2\lambda_1x+\lambda_2=0 & ① \\ L_y=xz+2\lambda_1y+\lambda_2=0 & ② \\ L_z=xy+2\lambda_1z+\lambda_2=0 & ③ \\ x^2+y^2+z^2=1 & ④ \\ xyz=0 & ⑤ \end{cases}$$

由①②③，两两作差：
$$\begin{cases} (2\lambda_1-z)(x-y)=0 \\ (2\lambda_1-y)(x-z)=0 \\ (2\lambda_1-x)(y-z)=0 \end{cases}$$

由④⑤，x,y,z 不可能同时相等.

∴ 1° $x=y\neq z$ 时

$$\begin{cases} 2x+z=0 \\ 2x^2+z^2=1 \end{cases} \Rightarrow \begin{cases} ①\ x=\frac{\sqrt6}{6},\ y=\frac{\sqrt6}{6},\ z=-\frac{\sqrt6}{3},\ \lambda_1=\frac{\sqrt6}{12},\ \lambda_2=\frac{1}{6},\quad f(X_1)=-\frac{\sqrt6}{18} \\ ②\ x=-\frac{\sqrt6}{6},\ y=-\frac{\sqrt6}{6},\ z=\frac{\sqrt6}{3},\ \lambda_1=-\frac{\sqrt6}{12},\ \lambda_2=\frac{1}{6},\quad f(X_2)=\frac{\sqrt6}{18} \end{cases}$$

2° $x=z\neq y$ 时

$$\begin{cases} 2x+y=0 \\ 2x^2+y^2=1 \end{cases} \Rightarrow \begin{cases} ③\ x=\frac{\sqrt6}{6},\ y=-\frac{\sqrt6}{3},\ z=\frac{\sqrt6}{6},\ \lambda_1=\frac{\sqrt6}{12},\ \lambda_2=\frac{1}{6},\quad f(X_3)=-\frac{\sqrt6}{18} \\ ④\ x=-\frac{\sqrt6}{6},\ y=\frac{\sqrt6}{3},\ z=-\frac{\sqrt6}{6},\ \lambda_1=-\frac{\sqrt6}{12},\ \lambda_2=\frac{1}{6},\quad f(X_4)=\frac{\sqrt6}{18} \end{cases}$$

3° $y=z\neq x$ 时

$$\begin{cases} 2y+x=0 \\ 2y^2+x^2=1 \end{cases} \Rightarrow \begin{cases} ⑤\ x=-\frac{\sqrt6}{3},\ y=\frac{\sqrt6}{6},\ z=\frac{\sqrt6}{6},\ \lambda_1=\frac{\sqrt6}{12},\ \lambda_2=\frac{1}{6},\quad f(X_5)=-\frac{\sqrt6}{18} \\ ⑥\ x=\frac{\sqrt6}{3},\ y=-\frac{\sqrt6}{6},\ z=-\frac{\sqrt6}{6},\ \lambda_1=-\frac{\sqrt6}{12},\ \lambda_2=\frac{1}{6},\quad f(X_6)=\frac{\sqrt6}{18} \end{cases}$$

容易验证：$(\frac{\sqrt6}{6},\frac{\sqrt6}{6},-\frac{\sqrt6}{3})$、$(\frac{\sqrt6}{6},-\frac{\sqrt6}{3},\frac{\sqrt6}{6})$、$(-\frac{\sqrt6}{3},\frac{\sqrt6}{6},\frac{\sqrt6}{6})$ 为极小值点，取得 min 值 $-\frac{\sqrt6}{18}$

$(-\frac{\sqrt6}{6},-\frac{\sqrt6}{6},\frac{\sqrt6}{3})$、$(-\frac{\sqrt6}{6},\frac{\sqrt6}{3},-\frac{\sqrt6}{6})$、$(\frac{\sqrt6}{3},-\frac{\sqrt6}{6},-\frac{\sqrt6}{6})$ 为极大值点，取得 max 值 $\frac{\sqrt6}{18}$

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3.

(1)

令 $\begin{cases} f_x=-6y-18x=0 \\ f_y=12-8y-6x=0 \end{cases} \Rightarrow \begin{cases} x_1=-\frac{2}{3} \\ y_1=2 \end{cases}$ 点 $P_1(-\frac{2}{3},2)$ 在区域 D 内部

P_1 是 D 内部唯一临界点；又 $f_{xx}=-18$，$f_{xy}=-6$，$f_{yy}=-8$，$H_f=\begin{pmatrix} -18 & -6 \\ -6 & -8 \end{pmatrix}$

$\because |H_f|>0$，$-18<0$ $\therefore f$ 在区域 D 内部取得极大值 $f(-\frac{2}{3},2)=12$.

在区域 D 的边界上，记 $L(x,y)=f(x,y)+\lambda(\frac{x^2}{4}+\frac{y^2}{9}-1)$

令 $\begin{cases} L_x=-6y-18x+\frac{\lambda}{2}x=0 \\ L_y=12-8y-6x+\frac{2}{9}\lambda y=0 \\ \frac{x^2}{4}+\frac{y^2}{9}=1 \end{cases} \Rightarrow \begin{cases} (\lambda-36)x-12y=0 \\ 54x+(72-2\lambda)y=108 \\ 9x^2+4y^2=36 \end{cases}$ 解得：

$\lambda_1=36$ $P_2(2,0)$ $f(P_2)=-36$

$\lambda_2=36-18\sqrt{3}$ $P_3(-1,\frac{3}{2}\sqrt{3})$ $f(P_3)=27\sqrt{3}-36$

$\lambda_3=36+18\sqrt{3}$ $P_4(-1,-\frac{3}{2}\sqrt{3})$ $f(P_4)=-27\sqrt{3}-36$

$H_L=\begin{pmatrix} \frac{\lambda}{2}-18 & -6 \\ -6 & \frac{2}{9}\lambda-8 \end{pmatrix}$ $|H_L|=36+\frac{1}{9}(\lambda-36)^2>0$，$\frac{\lambda}{2}-18>0 \Leftrightarrow \lambda>36$

$H_L(\lambda_2)<0$ $H_L(\lambda_3)>0$ $\therefore f(P_3)$ 为极大值，$f(P_4)$ 为极小值.

$\because -36-27\sqrt{3}<-36<12$，$\therefore f(P_2)$ 不是 max/min

经过大小比较，

$f_{max}=f(-\frac{2}{3},2)=12$

$f_{min}=f(-1,-\frac{3}{2}\sqrt{3})=-27\sqrt{3}-36$

3. (2)

$f(x,y,z) \le x+y+1 \le \sqrt{2(x^2+y^2)}+1 \le \sqrt{2z}+1 = 1+\sqrt{2}$ （$x=y=\frac{\sqrt{2}}{2}$，$z=1$ 时"="成立）

$f(x,y,z) \ge x+y+x^2+y^2 = (x+\frac{1}{2})^2+(y+\frac{1}{2})^2-\frac{1}{2} \ge -\frac{1}{2}$ （$x=-\frac{1}{2}$，$y=-\frac{1}{2}$，$z=\frac{1}{2}$ 时"="成立）

$\therefore f_{max}=1+\sqrt{2}$ $\quad f_{min}=-\frac{1}{2}$

(3). 在D内部：

$$\begin{cases} f_x = a-\dfrac{x}{\sqrt{1-x^2-y^2}}=0 \\ f_y = b-\dfrac{y}{\sqrt{1-x^2-y^2}}=0 \end{cases}$$

解得 $X_1\left(\frac{a}{\sqrt{a^2+b^2+1}},\frac{b}{\sqrt{a^2+b^2+1}}\right)$ $X_2\left(\frac{a}{\sqrt{a^2+b^2+1}},\frac{-b}{\sqrt{a^2+b^2+1}}\right)$ $X_3\left(\frac{-a}{\sqrt{a^2+b^2+1}},\frac{b}{\sqrt{a^2+b^2+1}}\right)$ $X_4\left(\frac{-a}{\sqrt{a^2+b^2+1}},\frac{-b}{\sqrt{a^2+b^2+1}}\right)$

$f(X_1)=\sqrt{a^2+b^2+1}$ $\quad f(X_2)=\frac{a^2-b^2+1}{\sqrt{a^2+b^2+1}}$ $\quad f(X_3)=\frac{b^2-a^2+1}{\sqrt{a^2+b^2+1}}$ $\quad f(X_4)=\frac{1-a^2-b^2}{\sqrt{a^2+b^2+1}}$

在D边界：$x^2+y^2=1$. 令 $x=\cos\theta$, $y=\sin\theta$

$z=a\cos\theta+b\sin\theta=\sqrt{a^2+b^2}\sin(\theta+\varphi)$

$\therefore -\sqrt{a^2+b^2} \le z \le \sqrt{a^2+b^2}$

在 $-\sqrt{a^2+b^2}$、$\sqrt{a^2+b^2}$、$f(X_1)$、$f(X_2)$、$f(X_3)$、$f(X_4)$ 六者中，最大者为 $\sqrt{a^2+b^2+1}$，最小者为 $-\sqrt{a^2+b^2}$

容易验证它们分别是极大/极小值，进而为最大/最小值.

4、

解：设 $f(x,y,z)=x^2+y^2+(z-c)^2$ 求 f 在条件 $z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ 下的最小值

记 $L(x,y,z)=x^2+y^2+(z-c)^2+\lambda\left(z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$

令
$$\begin{cases} L_x=2x-\frac{2\lambda}{a^2}x=0 \\ L_y=2y-\frac{2\lambda}{b^2}y=0 \\ L_z=2(z-c)+2\lambda z=0 \\ z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2} \end{cases}$$

解得 $X_1\left(0,\frac{\pm bc}{1+b^2},\frac{c}{1+b^2}\right)$

$X_2\left(\frac{\pm ac}{1+a^2},0,\frac{c}{1+a^2}\right)$

$f(X_1)=\frac{b^2c^2}{(b^2+1)^2}+\frac{c^2b^4}{(b^2+1)^2}=\frac{b^2c^2}{1+b^2}$ $\quad f(X_2)=\frac{a^2c^2}{1+a^2}$

$\because 0<b<a$

$\therefore f(X_1)$ 为极小值 (min) $f(X_2)$ 为极大值.（经检验）

习题11-A

1. $\frac{\partial u}{\partial x}=\frac{1}{1+\left(\frac{x^2+y^2}{x-y}\right)^2}\cdot\frac{2x(x-y)-(x^2+y^2)}{(x-y)^2}=\frac{x^2-y^2-2xy}{(x-y)^2+(x^2+y^2)^2}$

$\frac{\partial u}{\partial y}=\frac{1}{1+\left(\frac{x^2+y^2}{x-y}\right)^2}\cdot\frac{2y(x-y)+x^2+y^2}{(x-y)^2}=\frac{x^2-y^2+2xy}{(x-y)^2+(x^2+y^2)^2}$

$\because \frac{\sin 2u}{2}=\sin u\cos u=\frac{\tan u}{1+\tan^2 u}=\frac{\frac{x^2+y^2}{x-y}}{1+\left(\frac{x^2+y^2}{x-y}\right)^2}=\frac{(x-y)(x^2+y^2)}{(x-y)^2+(x^2+y^2)^2}$

而 $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x(x^2-y^2-2xy)+y(x^2-y^2+2xy)}{(x-y)^2+(x^2+y^2)^2}=\frac{x^3+xy^2-x^2y-y^3}{(x-y)^2+(x^2+y^2)^2}=\frac{(x-y)(x^2+y^2)}{(x-y)^2+(x^2+y^2)^2}$

$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{\sin 2u}{2}$

2. $\because u(r\cos\theta, r\sin\theta)=S(r)$ 与 θ 无关 $\therefore \frac{\partial u}{\partial \theta}=0$

$\therefore \frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}=-r\sin\theta f'(r\cos\theta)+r\cos\theta g'(r\sin\theta)=0$

即: $yf'(x)=xg'(y)$ 恒成立 即 $\frac{f'(x)}{x}=\frac{g'(y)}{y}$ 恒成立.

由 x, y 任意性 $\frac{f'(x)}{x}=\frac{g'(y)}{y}=c$

$\therefore f'(x)=c\cdot x$, $g'(y)=c\cdot y$ $\therefore f(x)=c'\cdot x^2+c''$, $g(y)=c'\cdot y^2+c'''$

$\therefore u(x,y)=f(x)+g(y)=C_1(x^2+y^2)+C_2$. ($C_1$、$C_2$ 为任意常数).

3. $\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=f'(r\cos\theta)g(r\sin\theta)\cdot\cos\theta+g'(r\sin\theta)\cdot f(r\cos\theta)\cdot\sin\theta=0.$

$\therefore xf'(x)\cdot g(y)+yg'(y)f(x)=0.$ 由 x,y 任意性（此式恒成立）：

$$\frac{xf'(x)}{f(x)}=-\frac{yg'(y)}{g(y)}=C$$

$\therefore \frac{f'(x)}{f(x)}=\frac{C}{x}\Rightarrow \ln|f(x)|=C\cdot\ln|x|\Rightarrow f(x)=C'\cdot x^{C}$

$\frac{g'(y)}{g(y)}=-\frac{C}{y}\Rightarrow \ln|g(y)|=-C\cdot\ln|y|\Rightarrow g(y)=C''\cdot(\frac{1}{y})^{C}$ $\Bigg\}\Rightarrow u(x,y)=C_1\cdot(\frac{x}{y})^{C_2}$

（C_1、C_2为任意常数）

4. 充分性：令 $g(\alpha)=\frac{f(\alpha x_1,\cdots,\alpha x_n)}{\alpha^{\lambda}}$

$\because \sum_{i=1}^{n}x_i\frac{\partial f}{\partial x_i}f(x_1,\cdots,x_n)=\lambda\cdot f(x_1,\cdots,x_n)$ $\therefore \alpha\cdot\sum_{i=1}^{n}x_i\cdot\frac{\partial f}{\partial x_i}(\alpha x_1,\cdots,\alpha x_n)=\lambda\cdot f(\alpha x_1,\cdots,\alpha x_n)$

$\therefore g'(\alpha)=\frac{1}{\alpha^{2\lambda}}\left[\alpha^{\lambda}\sum_{i=1}^{n}x_i\cdot\frac{\partial f}{\partial x_i}(\alpha x_1,\cdots,\alpha x_n)-\lambda\alpha^{\lambda-1}\cdot f(\alpha x_1,\cdots,\alpha x_n)\right]=0$

$\therefore g(\alpha)$ 恒为常数. $\because g(1)=f(x_1,\cdots,x_n)$ $\therefore g(\alpha)\equiv f(x_1,\cdots,x_n)$. f是λ次正齐次函数.

必要性：

$\because f(\alpha x_1,\cdots\alpha x_n)=\alpha^{\lambda}f(x_1,\cdots,x_n)$. 两边对$\alpha$求导

$\therefore \sum_{i=1}^{n}x_i\cdot\frac{\partial f}{\partial x_i}(\alpha x_1,\cdots,\alpha x_n)=\lambda\alpha^{\lambda-1}\cdot f(x_1,\cdots,x_n)$

令 $\alpha=1$ $\therefore \sum_{i=1}^{n}x_i\cdot\frac{\partial f}{\partial x_i}(x_1,\cdots,x_n)=\lambda\cdot f(x_1,\cdots,x_n)$

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5. $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x' \cos\theta + f_y' \sin\theta$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta f_x' + r\cos\theta f_y'$

$\frac{\partial u}{\partial r^2} = (f_{xx}'\cdot\cos\theta + f_{xy}'\sin\theta)\cos\theta + (f_{xy}'\cos\theta + f_{yy}'\sin\theta)\sin\theta = \cos^2\theta f_{xx} + \sin^2\theta f_{yy} + \sin 2\theta f_{xy}$

$\frac{\partial u}{\partial \theta^2} = -r\left[\cos\theta\cdot f_x' + \sin\theta\cdot(-r\sin\theta f_{xx}' + r\cos\theta f_{xy}')\right] + r\left[-\sin\theta f_y' + \cos\theta(-r\sin\theta f_{xy}' + r\cdot\cos\theta f_{yy}')\right]$

$= r^2\sin^2\theta\cdot f_{xx} + r^2\cos^2\theta f_{yy} - r^2\sin 2\theta f_{xy} - r\cos\theta f_x - r\sin\theta f_y$

$= r^2\sin^2\theta f_{xx} + r^2\cos^2\theta f_{yy} - r^2\sin 2\theta f_{xy} - r\cdot\frac{\partial u}{\partial r}$

$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = (\cos^2\theta + \sin^2\theta) f_{xx} + (\cos^2\theta + \sin^2\theta) f_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

6. 令 $s = \frac{x}{x^2+y^2}$, $t = \frac{y}{x^2+y^2}$ $\quad\therefore u(x,y) = f(s(x,y), t(x,y))$. $\quad f \begin{cases} s \begin{cases} x \\ y \end{cases} \\ t \begin{cases} x \\ y \end{cases} \end{cases}$

$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} = f_s'\cdot\frac{y^2-x^2}{(x^2+y^2)^2} - f_t'\cdot\frac{2xy}{(x^2+y^2)^2}$ $\qquad \frac{\partial u}{\partial y} = f_t'\cdot\frac{x^2-y^2}{(x^2+y^2)^2} - f_s'\cdot\frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial x^2} = \left(f_{ss}\cdot\frac{y^2-x^2}{(x^2+y^2)^2} + \cancel{f_{st}\cdot\frac{2xy}{(x^2+y^2)^2}}\right)\cdot\frac{y^2-x^2}{(x^2+y^2)^2} + f_s\cdot\frac{2x(x^2-3y^2)}{(x^2+y^2)^3} - \left(\cancel{f_{st}\cdot\frac{y^2-x^2}{(x^2+y^2)^2}} - f_{tt}\cdot\frac{2xy}{(x^2+y^2)^2}\right)\frac{2xy}{(x^2+y^2)^2} - f_t'\cdot\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$

$\frac{\partial^2 u}{\partial y^2} = -\left(f_{ss}\cdot\frac{-2xy}{(x^2+y^2)^2} + \cancel{f_{st}\cdot\frac{x^2-y^2}{(x^2+y^2)^2}}\right)\cdot\frac{2xy}{(x^2+y^2)^2} - f_s\cdot\frac{2x(x^2-3y^2)}{(x^2+y^2)^3} + \left(\cancel{f_{st}\cdot\frac{-2xy}{(x^2+y^2)^2}} + f_{tt}\cdot\frac{x^2-y^2}{(x^2+y^2)^2}\right)\frac{x^2-y^2}{(x^2+y^2)^2} + f_t'\cdot\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{(x^2-y^2)^2 + 4x^2y^2}{(x^2+y^2)^4} f_{ss} + \frac{(x^2-y^2)^2 + 4x^2y^2}{(x^2+y^2)^4} f_{tt} = \frac{1}{(x^2+y^2)^2}\left(\frac{\partial^2 f}{\partial s^2} + \frac{\partial^2 f}{\partial t^2}\right) = 0$

$\therefore u$ 也是调和函数.

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7. 令 $u=xy$, $v=\frac{x^2-y^2}{2}$. $g(x,y)=f(u,v)$

$$\therefore \frac{\partial g}{\partial x}=\frac{\partial f}{\partial u}\cdot y+\frac{\partial f}{\partial v}\cdot x \qquad \frac{\partial g}{\partial y}=\frac{\partial f}{\partial u}\cdot x-\frac{\partial f}{\partial v}\cdot y$$

$$\therefore \frac{\partial^2 f}{\partial x^2}=\frac{\partial\left(\frac{\partial f}{\partial u}\right)}{\partial u}\cdot\frac{\partial u}{\partial x}\cdot y+\frac{\partial\left(\frac{\partial f}{\partial u}\right)}{\partial v}\cdot\frac{\partial v}{\partial x}\cdot x+\frac{\partial f}{\partial v}=\frac{\partial^2 f}{\partial u^2}y^2+\frac{\partial^2 f}{\partial v^2}x^2+\frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial\left(\frac{\partial f}{\partial u}\right)}{\partial u}\cdot\frac{\partial u}{\partial y}\cdot x-\frac{\partial\left(\frac{\partial f}{\partial v}\right)}{\partial v}\cdot\frac{\partial v}{\partial y}\cdot y-\frac{\partial f}{\partial v}=\frac{\partial^2 f}{\partial u^2}\cdot x^2+\frac{\partial^2 f}{\partial v^2}y^2-\frac{\partial f}{\partial v}$$

$$\therefore \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)(x^2+y^2)=x^2+y^2$$

8. $$\frac{\partial z}{\partial x}=\varphi\left(\frac{y}{x}\right)-\frac{y}{x}\cdot\varphi'\left(\frac{y}{x}\right)-\frac{y}{x^2}\psi'\left(\frac{y}{x}\right) \qquad \frac{\partial z}{\partial y}=\varphi'\left(\frac{y}{x}\right)+\frac{1}{x}\psi'\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial^2 z}{\partial x^2}=\cancel{-\frac{y}{x^2}\varphi'\left(\frac{y}{x}\right)}+\cancel{\frac{y}{x^2}\varphi'\left(\frac{y}{x}\right)}+\frac{y^2}{x^3}\varphi''\left(\frac{y}{x}\right)+\frac{2y}{x^3}\psi'\left(\frac{y}{x}\right)+\frac{y^2}{x^4}\psi''\left(\frac{y}{x}\right)$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{1}{x}\varphi''\left(\frac{y}{x}\right)+\frac{1}{x^2}\psi''\left(\frac{y}{x}\right) \qquad \frac{\partial^2 z}{\partial x\partial y}=-\frac{y}{x^2}\varphi''\left(\frac{y}{x}\right)-\frac{1}{x^2}\psi'\left(\frac{y}{x}\right)-\frac{y}{x^3}\psi''\left(\frac{y}{x}\right)$$

$$\therefore x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\varphi''+\frac{2y}{x}\psi'+\frac{y^2}{x^2}\psi''+\frac{y^2}{x}\varphi''+\frac{y^2}{x^2}\psi''-\frac{2y^2}{x}\varphi''-\frac{2y}{x}\psi'-\frac{2y^2}{x^2}\psi''=0$$

9. 设 $X_0=(x_{10},x_{20},\cdots,x_{n0})$ $Y_0=(y_{10},y_{20},\cdots,y_{n0})$ $\therefore \varphi_Y(t)=f(x_{10}+ty_{10},\cdots,x_{n0}+ty_{n0})$

$$\therefore \varphi_Y'(t)=y_{10}\cdot f'_{x_1}+y_{20}\cdot f'_{x_2}+\cdots+y_{n0}f'_{x_n}$$

$$\therefore \varphi_Y''(t)=y_{10}(y_{10}f''_{x_1x_1}+y_{20}f''_{x_1x_2}+\cdots+y_{n0}f''_{x_1x_n})+\cdots+y_{n0}(y_{10}f''_{x_nx_1}+\cdots\cdots+y_{n0}f''_{x_nx_n})$$

$\therefore \varphi_Y''(t)=Y_0'\cdot H_f(X_0)\cdot Y_0>0$. 由 Y_0 任意性知 $H_f(X_0)$ 正定.

10. 使用数学归纳法证明：

$n=1$时. $\because \frac{\partial u}{\partial y}=\varphi'(z)\cdot\frac{x}{(1-y)^2}$, $\frac{\partial u}{\partial x}=\varphi'(z)\cdot\frac{1}{1-y}$

$\therefore \frac{\partial u}{\partial y}=\frac{x}{1-y}\cdot\frac{\partial u}{\partial x}=z\cdot\frac{\partial u}{\partial x}$ 成立.

$n=k-1$时. 假设 $\frac{\partial^{k-1}u}{\partial y^{k-1}}=\frac{\partial^{k-2}}{\partial x^{k-2}}\left(z^{k-1}\cdot\frac{\partial u}{\partial x}\right)$ 成立.

则$n=k$时. $\frac{\partial^k u}{\partial y^k}=\frac{\partial}{\partial y}\left(\frac{\partial^{k-1}u}{\partial y^{k-1}}\right)=\frac{\partial}{\partial y}\frac{\partial^{k-2}}{\partial x^{k-2}}\left(z^{k-1}\frac{\partial u}{\partial x}\right)=\frac{\partial^{k-2}}{\partial x^{k-2}}\frac{\partial}{\partial y}\left(z^{k-1}\cdot\frac{\partial u}{\partial x}\right)$

$$=\frac{\partial^{k-2}}{\partial x^{k-2}}\left((k-1)z^{k-2}\cdot\frac{x}{(1-y)^2}\cdot\frac{\partial u}{\partial x}+z^{k-1}\cdot\frac{\partial^2 u}{\partial y\partial x}\right)$$

$$\frac{\partial^{k-1}}{\partial x^{k-1}}\left(z^k\frac{\partial u}{\partial x}\right)=\frac{\partial^{k-1}}{\partial x^{k-1}}\left(z^{k-1}\cdot\frac{\partial u}{\partial y}\right)=\frac{\partial^{k-2}}{\partial x^{k-2}}\cdot\frac{\partial}{\partial x}\left(z^{k-1}\cdot\frac{\partial u}{\partial y}\right)$$

$$=\frac{\partial^{k-2}}{\partial x^{k-2}}\left((k-1)z^{k-2}\cdot\frac{1}{1-y}\cdot\frac{\partial u}{\partial y}+z^{k-1}\cdot\frac{\partial^2 u}{\partial x\partial y}\right)$$ 对比两式：

$\because \frac{\partial u}{\partial y}=z\cdot\frac{\partial u}{\partial x}$ 且 $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$. $\therefore \frac{\partial^k u}{\partial y^k}=\frac{\partial^{k-1}}{\partial x^{k-1}}\left(z^k\cdot\frac{\partial u}{\partial x}\right)$.

<注> Lagrange公式：$u=\varphi(z)$. $z=(x,y)$是方程$z=x+y\cdot f(z)$确定的隐函数. 则$\frac{\partial^n u}{\partial y^n}=\frac{\partial^{n-1}}{\partial x^{n-1}}\left(z^n\frac{\partial u}{\partial x}\right)$

11. 必要性：$\because f(x,y)=g(x)h(y)$ $\therefore f_x=g'(x)h(y)$ $f_y=g(x)h'(y)$ $f_{xy}=g'(x)h'(y)$

$\therefore f\cdot f_{xy}=f_x\cdot f_y=g(x)h(y)g'(x)h'(y)$

充分性：令$F=\frac{f_x}{f}$ ($f\neq 0$) $\therefore F_y=\frac{f\cdot f_{xy}-f_x f_y}{f^2}=0$

$\therefore F$与y无关 $\therefore$ 记$F=P(x)=\frac{f_x}{f}$ $\therefore f_{xy}=f_y\cdot P(x)$

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13.

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\lim_{\Delta y\to 0}\frac{1}{\Delta y}\Big(f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)-f(x_0,y_0+\Delta y)+f(x_0,y_0)\Big)\right]$$

令 $\varphi(x)=f(x,y_0+\Delta y)-f(x,y_0)$

$$\therefore \frac{1}{\Delta x\Delta y}\Big[\big(f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)\big)-\big(f(x_0,y_0+\Delta y)-f(x_0,y_0)\big)\Big]\triangleq \Phi$$

$$=\frac{1}{\Delta x\Delta y}\left[\varphi(x_0+\Delta x)-\varphi(x_0)\right]=\frac{1}{\Delta y}\varphi'(x_0+\theta\Delta x)=\frac{1}{\Delta y}\left[f_x(x_0+\theta\Delta x,y_0+\Delta y)-f_x(x_0+\theta\Delta x,y_0)\right]$$

$(0<\theta<1)$

因为 f_x 在 (x_0,y_0) 可微

$$\therefore f_x(x_0+\theta\Delta x,y_0+\Delta y)-f_x(x_0,y_0)=\Delta f_x=f_{xx}(x_0,y_0)\cdot\theta\Delta x+f_{xy}(x_0,y_0)\Delta y+o_1\left(\sqrt{\Delta x^2+\Delta y^2}\right)$$

$$f_x(x_0+\theta\Delta x,y_0)-f_x(x_0,y_0)=\Delta f_x=f_{xx}(x_0,y_0)\,\theta\Delta x+o_2(\theta\Delta x)$$

$$\Phi=\frac{1}{\Delta y}\left[f_{xy}(x_0,y_0)\cdot\Delta y+o_1-o_2\right]=f_{xy}(x_0,y_0)+o_1-o_2$$

同理 $\Phi=f_{yx}(x_0,y_0)+o_1-o_2$

$$\therefore \lim_{\Delta x\to 0}\lim_{\Delta y\to 0}\Phi=f_{xy}=f_{yx}\Big|_{x_0,y_0}$$

14. 要证 $f_{xy}(x,y)$ 在 (x_0,y_0) 处连续、 只需证 $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f_{xy}(x,y) = f_{xy}(x_0,y_0)$.

令 $\Delta y = y-y_0$ ∵ $f_{xy}(x_0,y_0)=\lim\limits_{\Delta y\to 0}\dfrac{f_x(x_0,y_0+\Delta y)-f_x(x_0,y_0)}{\Delta y}$

∵ f_x 存在于 $U(M)$ 内 由中值定理. $f_x(x_0,y_0+\Delta y)-f_x(x_0,y_0)=\Delta y\cdot f_{xy}(x_0,y_0+\theta\Delta y)$

∴ 上式 $=\lim\limits_{\Delta y\to 0} f_{xy}(x_0,y_0+\theta\Delta y)=\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f_{xy}(x,y)$

∴ $f_{xy}(x,y)$ 在 (x_0,y_0) 连续 由P49定理 $f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$.

15、 **本题的实质是关于复合函数可微性的相关讨论。**
以下内容节选自华师大《数学分析（下册）》p118-119.
如果证明了这个定理，那么本题结论的成立是自然的。抑或，可以仿照下述的证明思路来给出一个证明。

定理 17.5 若函数 $x=\varphi(s,t)$，$y=\psi(s,t)$ 在点 $(s,t)\in D$ 可微，$z=f(x,y)$ 在点 $(x,y)=(\varphi(s,t),\psi(s,t))$ 可微，则复合函数

$$z=f(\varphi(s,t),\psi(s,t))$$

在点 (s,t) 可微，且它关于 s 与 t 的偏导数分别为

$$\begin{aligned}\left.\frac{\partial z}{\partial s}\right|_{(s,t)}&=\left.\frac{\partial z}{\partial x}\right|_{(x,y)}\left.\frac{\partial x}{\partial s}\right|_{(s,t)}+\left.\frac{\partial z}{\partial y}\right|_{(x,y)}\left.\frac{\partial y}{\partial s}\right|_{(s,t)},\\ \left.\frac{\partial z}{\partial t}\right|_{(s,t)}&=\left.\frac{\partial z}{\partial x}\right|_{(x,y)}\left.\frac{\partial x}{\partial t}\right|_{(s,t)}+\left.\frac{\partial z}{\partial y}\right|_{(x,y)}\left.\frac{\partial y}{\partial t}\right|_{(s,t)}.\end{aligned}\tag{4}$$

证 由假设 $x=\varphi(s,t)$，$y=\psi(s,t)$ 在点 (s,t) 可微，于是

$$\Delta x=\frac{\partial x}{\partial s}\Delta s+\frac{\partial x}{\partial t}\Delta t+\alpha_1\Delta s+\beta_1\Delta t,\tag{5}$$

$$\Delta y=\frac{\partial y}{\partial s}\Delta s+\frac{\partial y}{\partial t}\Delta t+\alpha_2\Delta s+\beta_2\Delta t,\tag{6}$$

其中当 $\Delta s,\Delta t$ 趋于零时，$\alpha_1,\alpha_2,\beta_1,\beta_2$ 都趋向于零. 又由 $z=f(x,y)$ 在点 (x,y) 可微，所以

$$\Delta z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y+\alpha\Delta x+\beta\Delta y,\tag{7}$$

其中当 $\Delta x,\Delta y\to 0$ 时，$\alpha,\beta\to 0$（我们补充 α,β 之定义使当 $\Delta x=0,\Delta y=0$ 时，$\alpha=\beta=0$），将(5)，(6)代入(7)，得

$$\begin{aligned}\Delta z=&\left(\frac{\partial z}{\partial x}+\alpha\right)\left(\frac{\partial x}{\partial s}\Delta s+\frac{\partial x}{\partial t}\Delta t+\alpha_1\Delta s+\beta_1\Delta t\right)\\&+\left(\frac{\partial z}{\partial y}+\beta\right)\left(\frac{\partial y}{\partial s}\Delta s+\frac{\partial y}{\partial t}\Delta t+\alpha_2\Delta s+\beta_2\Delta t\right).\end{aligned}$$

整理后

$$\Delta z=\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)\Delta s+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)\Delta t+\bar{\alpha}\Delta s+\bar{\beta}\Delta t,\tag{8}$$

其中

$$\bar{\alpha}=\frac{\partial z}{\partial x}\alpha_1+\frac{\partial z}{\partial y}\alpha_2+\frac{\partial x}{\partial s}\alpha+\frac{\partial y}{\partial s}\beta+\alpha\alpha_1+\beta\alpha_2,\tag{9}$$

$$\bar{\beta}=\frac{\partial z}{\partial x}\beta_1+\frac{\partial z}{\partial y}\beta_2+\frac{\partial x}{\partial t}\alpha+\frac{\partial y}{\partial t}\beta+\alpha\beta_1+\beta_1\beta_2.\tag{10}$$

由于 $\varphi(s,t),\psi(s,t)$ 在点 (s,t) 可微，因此它们在点 (s,t) 都连续，即当 Δs，$\Delta t\to 0$时，有 $\Delta x,\Delta y\to 0$. 从而也有 $\alpha\to 0,\beta\to 0$，以及 $\alpha_1,\alpha_2,\beta_1,\beta_2\to 0$. 于是在(9)、(10)式中，当 $\Delta s,\Delta t\to 0$,，有 $\bar{\alpha}\to 0,\bar{\beta}\to 0$. 故由(8)式推得复合函数(3)可微并求得 z 关于 s 和 t 的偏导数(4).

这里公式(4)也称为**链式法则**.

16. ∵ F 在 X_0 可微 ∴ $F(X)-F(X_0)=J_F(X_0)\cdot\Delta X+o(\|\Delta X\|)=\left(J_F(X_0)+o\left(\frac{\|\Delta X\|}{\Delta X}\right)\right)\cdot\Delta X$

其中 $\Delta X=X-X_0$. 由 $\lim\limits_{\Delta X\to 0}o\left(\frac{\|\Delta X\|}{\Delta X}\right)=0$ 知：存在 X_0 的邻域 U，使：

当 $X\in U$ 时 $\left|o\left(\frac{\|\Delta X\|}{\Delta X}\right)\right|\leq 1$. 进而 $\|F(x)-F(X_0)\|\leq(J_F(X_0)+1)\cdot\|\Delta X\|$.

17. $\frac{\partial u}{\partial \vec{l}}=\frac{\partial f}{\partial x}\cdot\cos\alpha+\frac{\partial f}{\partial y}\cdot\cos\beta+\frac{\partial f}{\partial z}\cdot\cos\gamma$

$\therefore \frac{\partial}{\partial\vec{l}}\left(\frac{\partial u}{\partial\vec{l}}\right)=\cos\alpha\cdot(\cos\alpha\cdot f_{xx}'+\cos\beta\cdot f_{yx}'+\cos\gamma\cdot f_{zx}')+\cos\beta\cdot(\cos\alpha\cdot f_{xy}'+\cos\beta\cdot f_{yy}'+\cos\gamma\cdot f_{zy}')$

$+\cos\gamma\,(\cos\alpha\cdot f_{xz}'+\cos\beta\cdot f_{yz}'+\cos\gamma\cdot f_{zz}')=\vec{l}\cdot H_f\cdot\vec{l}^{\,T}$

18. (1) $\frac{\partial f}{\partial\vec{l}}(x)=\langle\nabla f(x),\vec{l}\rangle=\langle\nabla f(x),\sum\limits_{i=1}^{n}\langle\vec{l},\vec{l_i}\rangle\vec{l_i}\rangle=\sum\limits_{i=1}^{n}\langle\vec{l},\vec{l_i}\rangle\cdot\langle\nabla f(x),\vec{l_i}\rangle=\sum\limits_{i=1}^{n}\langle\vec{l},\vec{l_i}\rangle\cdot\frac{\partial f}{\partial\vec{l_i}}(x)$

(2) $\sum\limits_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2=\sum\limits_{i=1}^{n}\langle\nabla f,\vec{e_i}\rangle^2=\sum\limits_{i=1}^{n}|\nabla f|^2\cdot\cos^2\alpha_i=|\nabla f|^2\cdot\sum\limits_{i=1}^{n}\cos^2\alpha_i=|\nabla f|^2\cdot\sum\limits_{i=1}^{n}\cos^2\beta_i=\sum\limits_{i=1}^{n}\left(\frac{\partial f}{\partial\vec{l_i}}\right)^2$

$\vec{e_i}=(0,0,\cdots,0,\underset{(i)}{1},0,\cdots 0)$ α_i：∇f 与 $\vec{e_i}$ 夹角 β_i：∇f 与 $\vec{l_i}$ 夹角.

(3) $\langle\nabla f,\vec{l_i}\rangle=\frac{\partial f}{\partial\vec{l_i}}$ $\nabla f=\sum\limits_{i=1}^{n}\langle\nabla f,\vec{l_i}\rangle=\sum\limits_{i=1}^{n}\frac{\partial f}{\partial\vec{l_i}}\cdot\vec{l_i}$

19. $(x,y)\in\partial D$ 时结论显然成立. 不妨设 $(x,y)\in D^\circ$. 令 $\vec{l}=(\cos\theta,\sin\theta)=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)$. $t=1-\sqrt{x^2+y^2}$

∵ D° 是凸区域. 由微分中值定理. ∃ $\xi\in(0,t)$ 使：

$f(x+t\cos\theta,y+t\sin\theta)-f(x,y)=t\cos\theta\cdot f_x'(x+\xi\cos\theta,y+\xi\sin\theta)+t\sin\theta\cdot f_y'(x+\xi\cos\theta,y+\xi\sin\theta)$

$=t\cdot\frac{\partial f}{\partial\vec{l}}(x+\xi\cos\theta,y+\xi\sin\theta)$.

由于 $\left|\frac{\partial f}{\partial\vec{l}}\right|\leq 1$ 且 $f(x+t\cos\theta,y+t\sin\theta)=0$. （注意到 $(x+t\cos\theta,y+t\sin\theta)\in\partial D$）

因此 $|f(x,y)|\leq t=1-\sqrt{x^2+y^2}$

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20. 对 $\forall X_1、X_2 \in D$. 设 $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \in [-M, M]$, $M \in \mathbb{R}^+$, $X_1(x_1,y_1)$ $X_2(x_2,y_2)$

由于f偏导存在. $\lim\limits_{t\to 0} \frac{f(X_1+t(X_2-X_1))-f(X_1)}{t} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$.

$\therefore 0 = \lim\limits_{t\to 0} -2Mt \le \lim\limits_{t\to 0} f(X_1+t(X_2-X_1)) - f(X_1) \le \lim\limits_{t\to 0} 2Mt = 0$

$\because$ D是凸域 $\therefore t\to 0$时. $X_3 = X_1 + t(X_2-X_1) \in D$

由于$X_1、X_2$任意性. 故X_3也有任意性

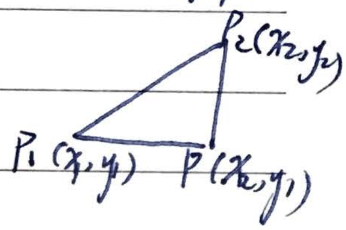
$\therefore$ 对 $\forall \varepsilon>0, \exists \delta>0$ s.t. $\forall X_1、X_3 \in D$: $|X_3-X_1|<\delta$, 总有 $|f(X_3)-f(X_1)|<\varepsilon$

$\therefore f(x,y)$ 在D上一致连续.

或:

2. f在$\mathbb{R}^2$上有连续偏导数 且 $|f_x(x,y)| \le M$ $|f_y(x,y)| \le M$

则对 $\forall P_1、P_2 \in \mathbb{R}^2$, 求证 $|f(P_1)-f(P_2)| \le \sqrt{2}M\,\|P_1-P_2\|$ $=\sqrt{2}M\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$



$|f(P_1)-f(P_2)| \le |f(P_1)-f(P)| + |f(P)-f(P_2)|$

由微分中值定理

$f_x|x_1-x_2| + f_y|y_1-y_2| \le M(|x_1-x_2|+|y_1-y_2|)$

$\le \sqrt{2}M\cdot\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

21. $D=\{(x,y)\in\mathbb{R}^2 \mid 1<x^2+y^2<4\}$ 为定义域

$$f(x,y)=\begin{cases} \arctan\frac{y}{x} & x>0,\ y>0 \\ \frac{\pi}{2} & x=0,\ y>0 \\ \pi+\arctan\frac{y}{x} & x<0 \\ \frac{3}{2}\pi & x=0,\ y<0 \\ 2\pi+\arctan\frac{y}{x} & x>0,\ y<0 \end{cases}$$

$f_x'(x,y) = -\frac{y}{x^2+y^2}$ $f_y'(x,y)=\frac{x}{x^2+y^2}$

$|f_x'| \le 2$, $|f_y'| \le 2$

若取 $x_n = \{(\frac{3}{2}, \frac{1}{n})\}$ $y_n = \{(\frac{3}{2}, -\frac{1}{n})\}$

则 $\lim\limits_{n\to\infty}|X_n - Y_n| = 0$ 但 $\lim\limits_{n\to\infty}|f(X_n)-f(Y_n)| = 2\pi > 0$ 不连续.

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22 由已知：$u^2+v^2=1$，$u'_x=v'_y$，$u'_y=-v'_x$.

$$\therefore \begin{cases} 2u\cdot u'_x+2v\cdot v'_x=0 \\ 2u\cdot u'_y+2v\cdot v'_y=0 \end{cases} \Rightarrow \begin{cases} u\cdot u'_x=-v\cdot v'_x \\ u\cdot u'_y=-v\cdot v'_y \end{cases} \Rightarrow \begin{cases} u\cdot u'_x=v\cdot u'_y \\ u\cdot v'_x=v\cdot v'_y \end{cases}$$

两边相乘 $u^2\cdot u'_x v'_x=v^2 u'_y v'_y=(1-u^2)u'_y v'_y=(u^2-1)u'_x v'_x$ $\therefore u'_x v'_x=0$.

u'_x，v'_x 中必有一者为零. 不妨设 $u'_x=0$. $\therefore v'_y=0$

$$\therefore \begin{cases} u\cdot u'_y=0 \\ v\cdot v'_x=0 \end{cases} \Rightarrow \because u^2+v^2=1$$ $\therefore u,v$ 不可能在 D 上同时恒为 0

$\therefore \exists u,v$ 中一者不为零. 不妨设 u 不恒为零 $\therefore u'_y=0$ $\therefore v'_x=0$

$\because u'_x=v'_x=u'_y=v'_y=0$ $\therefore u,v$ 在 D 上恒为常数.

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23. 题目修正为 $f(x,y)$ 在 $(0,0)$ 的邻域内二阶偏导数连续.（单点不能展开）

由多元 taylor 公式

$$f(2h,e^{-\frac{1}{2h}})=f(0,0)+\left[f_x'(0,0)\cdot 2h+f_y'(0,0)\cdot e^{-\frac{1}{2h}}\right] \qquad 0<\theta_1<1$$

$$+\frac{1}{2}\left[f_{xx}''(2\theta_1 h,\theta_1 e^{-\frac{1}{2h}})\cdot 4h^2+2f_{xy}''(2\theta_1 h,\theta_1 e^{-\frac{1}{2h}})\cdot 2he^{-\frac{1}{2h}}+f_{yy}''(\underbrace{2\theta_1 h}_{=\xi_1},\underbrace{\theta_1 e^{-\frac{1}{2h}}}_{=\eta_1})\cdot e^{-\frac{1}{h}}\right]$$

$$f(h,e^{-\frac{1}{h}})=f(0,0)+\left[f_x'(0,0)\cdot h+f_y'(0,0)\cdot e^{-\frac{1}{h}}\right] \qquad 0<\theta_2<1$$

$$+\frac{1}{2}\left[f_{xx}''(\theta_2 h,\theta_2 e^{-\frac{1}{h}})\cdot h^2+2f_{xy}''(\theta_2 h,\theta_2 e^{-\frac{1}{h}})\cdot he^{-\frac{1}{h}}+f_{yy}''(\underbrace{\theta_2 h}_{=\xi_2},\underbrace{\theta_2 e^{-\frac{1}{h}}}_{=\eta_2})\cdot e^{-\frac{2}{h}}\right]$$

$$\therefore f(2h,e^{-\frac{1}{2h}})-2f(h,e^{-\frac{1}{h}})+f(0,0)$$

$$=f_y'(0,0)\cdot\underbrace{(e^{-\frac{1}{2h}}-2e^{-\frac{1}{h}})}_{\div h^2\text{后}\to 0}+2h^2f_{xx}''(\xi_1,\eta_1)-h^2f_{xx}''(\xi_2,\eta_2)$$

由于 $\lim\limits_{h\to 0^+}\frac{e^{-\frac{1}{h}}}{h^2}=\lim\limits_{x\to+\infty}\frac{x^2}{e^x}=0$.

$$+2h\underbrace{\left[e^{-\frac{1}{2h}}f_{xy}''(\xi_1,\eta_1)-e^{-\frac{1}{h}}f_{xy}''(\xi_2,\eta_2)\right]}_{\div h^2\text{后}\to 0}+\underbrace{\left[f_{yy}''(\xi_1,\eta_1)\cdot e^{-\frac{1}{h}}-f_{yy}''(\xi_2,\eta_2)\cdot e^{-\frac{2}{h}}\right]}_{\div h^2\text{后}\to 0}$$

$$\therefore \lim_{h\to 0^+}\frac{f(2h,e^{-\frac{1}{2h}})-2f(h,e^{-\frac{1}{h}})+f(0,0)}{h^2}=f_{xx}''(0,0)$$

24. 记 $f(x,y)=x-y-\varphi(y)$ $\quad\because \varphi(0)=0 \quad \therefore f(0,0)=0$

$\because \varphi'(y)\in C(-a,a)$ $\quad\therefore f(x,y)\in C\,B_\delta(0)$, $f_x=1\in C\,B_\delta(0)$, $f_y=-1-\varphi'(y)\in C\,B_\delta$

且 $f_y(0,0)=-1-\varphi'(0)<0$.

$$\left.\begin{array}{l} f(0,0)=0 \\ f_y(0,0)\neq 0 \\ f\text{、}f_x\text{、}f_y\text{ 在}(0,0)\text{连续}\end{array}\right\}\Rightarrow$$

由隐函数存在定理 $\exists\,\delta>0$.

当 $|x|<\delta$ 时 有唯一可微函数 $y=y(x)$

满足 $x=y(x)+\varphi(y(x))$ 且 $y(0)=0$

A-25

1) 对u求偏导: $\Rightarrow \begin{cases} 2x\cdot\frac{\partial x}{\partial u}-\frac{\partial y}{\partial u}\cos(uv)+yv\sin(uv)+2z\frac{\partial z}{\partial u}=0 \\ 2x\cdot\frac{\partial x}{\partial u}+2y\cdot\frac{\partial y}{\partial u}-v\cos(uv)+4z\frac{\partial z}{\partial u}=0 \\ \frac{\partial x}{\partial u}y+x\frac{\partial y}{\partial u}-\cos u\cos v+\frac{\partial z}{\partial u}=0 \end{cases}$

当 $x=y=1$, $z=0$. $u=\frac{\pi}{2}$, $v=0$ 时.

代入得 $\Rightarrow \begin{cases} 2\frac{\partial x}{\partial u}\Big|_{(\frac{\pi}{2},0)}-\frac{\partial y}{\partial u}\Big|_{(\frac{\pi}{2},0)}=0 \\ \frac{\partial x}{\partial u}\Big|_{(\frac{\pi}{2},0)}+\frac{\partial y}{\partial u}\Big|_{(\frac{\pi}{2},0)}=0 \\ \frac{\partial x}{\partial u}\Big|_{(\frac{\pi}{2},0)}+\frac{\partial y}{\partial u}\Big|_{(\frac{\pi}{2},0)}+\frac{\partial z}{\partial u}\Big|_{(\frac{\pi}{2},0)}=0 \end{cases}$ 解得 $\Rightarrow \frac{\partial x}{\partial u}\Big|_{(\frac{\pi}{2},0)}=0$

2) 对v求偏导 $\Rightarrow \begin{cases} 2x\frac{\partial x}{\partial v}-\frac{\partial y}{\partial v}\cos(uv)+yu\sin(uv)+2z\frac{\partial z}{\partial v}=0 \\ 2x\frac{\partial x}{\partial v}+2y\cdot\frac{\partial y}{\partial v}-u\cos(uv)+4z\frac{\partial z}{\partial v}=0 \\ y\frac{\partial x}{\partial v}+x\frac{\partial y}{\partial v}+\sin u\cdot\sin v+\frac{\partial z}{\partial v}=0 \end{cases}$

$p_0(1,1,0,\frac{\pi}{2},0)$

代入得 $\Rightarrow \begin{cases} 2\frac{\partial x}{\partial v}\Big|_{p_0}-\frac{\partial y}{\partial v}\Big|_{p_0}=0 \\ 2\frac{\partial x}{\partial v}\Big|_{p_0}+2\frac{\partial y}{\partial v}\Big|_{p_0}-\frac{\pi}{2}=0 \\ \frac{\partial x}{\partial v}\Big|_{p_0}+\frac{\partial y}{\partial v}\Big|_{p_0}+\frac{\partial z}{\partial v}\Big|_{p_0}=0 \end{cases}$ 解得 $\Rightarrow \frac{\partial x}{\partial v}\Big|_{p_0}=\frac{\pi}{12}$

26.

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{dr}{dt}\cos\theta - r\sin\theta\dfrac{d\theta}{dt} \\ \dfrac{dy}{dt} = \dfrac{dr}{dt}\sin\theta + r\cos\theta\dfrac{d\theta}{dt} \end{cases}$$

又∵ $\begin{cases} x^2+y^2=r^2 \\ x=r\cos\theta \\ y=r\sin\theta \end{cases}$

∴ 原方程组可化为 $\Rightarrow \begin{cases} \dfrac{dr}{dt}\cos\theta - r\sin\theta\dfrac{d\theta}{dt} = r\sin\theta + kr^3\cos\theta \\ \dfrac{dr}{dt}\sin\theta + r\cos\theta\dfrac{d\theta}{dt} = -r\cos\theta + kr^3\sin\theta \end{cases} \Rightarrow \begin{cases} \dfrac{dr}{d\theta} = kr^3 \\ \dfrac{d\theta}{dt} = -1 \end{cases}$

27.

∵ $ze^z = xe^x + ye^y$

① $\dfrac{\partial z}{\partial x}\cdot e^z + ze^z\cdot\dfrac{\partial z}{\partial x} = (1+x)e^x$

∴ $\dfrac{\partial z}{\partial x} = \dfrac{(1+x)e^x}{(1+z)e^z}$

∴ $\dfrac{\partial u}{\partial x} = \dfrac{(1+\frac{\partial z}{\partial x})(y+z) - (x+z)\frac{\partial z}{\partial x}}{(y+z)^2} = \dfrac{(y-x)\frac{\partial z}{\partial x}}{(y+z)^2} + \dfrac{1}{y+z} = \dfrac{(y-x)(1+x)e^x}{(y+z)^2(1+z)e^z} + \dfrac{1}{y+z}$

② $\dfrac{\partial z}{\partial y}e^z + z\cdot e^z\cdot\dfrac{\partial z}{\partial y} = (1+y)e^y$

∴ $\dfrac{\partial z}{\partial y} = \dfrac{(1+y)e^y}{(1+z)e^z}$

∴ $\dfrac{\partial u}{\partial y} = \dfrac{\frac{\partial z}{\partial y}(y+z) - (x+z)(1+\frac{\partial z}{\partial y})}{(y+z)^2} = \dfrac{(y-x)\frac{\partial z}{\partial y}}{(y+z)^2} - \dfrac{x+z}{(y+z)^2} = \dfrac{(y-x)(1+y)e^y}{(y+z)^2(1+z)e^z} - \dfrac{x+z}{(y+z)^2}$

28. 对三个方程分别对x，对y，对z求偏导。

对x求：
$$\begin{cases} 1 = f_u \dfrac{\partial u}{\partial x} + f_v \dfrac{\partial v}{\partial x} + f_w \dfrac{\partial w}{\partial x} \\ 0 = g_u \dfrac{\partial u}{\partial x} + g_v \dfrac{\partial v}{\partial x} + g_w \dfrac{\partial w}{\partial x} \\ 0 = h_u \dfrac{\partial u}{\partial x} + h_v \dfrac{\partial v}{\partial x} + h_w \dfrac{\partial w}{\partial x} \end{cases}$$

记 $D = \begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}$

$D_1 = \begin{vmatrix} g_v & g_w \\ h_v & h_w \end{vmatrix}$ $D_2 = \begin{vmatrix} f_v & f_w \\ h_v & h_w \end{vmatrix}$ $D_3 = \begin{vmatrix} f_v & f_w \\ g_v & g_w \end{vmatrix}$

由Crammer法则
$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} 1 & f_v & f_w \\ 0 & g_v & g_w \\ 0 & h_v & h_w \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = \frac{\begin{vmatrix} g_v & g_w \\ h_v & h_w \end{vmatrix}}{\begin{vmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{vmatrix}} = \frac{D_1}{D}$$
同理 $\dfrac{\partial u}{\partial y} = -\dfrac{D_2}{D}$ $\dfrac{\partial u}{\partial z} = \dfrac{D_3}{D}$

29. 三个方程分别对x求导

（注意u是关于x的一元函数，故y、z均是x的函数）

$$\begin{cases} \dfrac{du}{dx} = f_x + f_y \dfrac{dy}{dx} + f_z \dfrac{dz}{dx} \\ 0 = g_x + g_y \dfrac{dy}{dx} + g_z \dfrac{dz}{dx} \\ 0 = h_x + h_y \dfrac{dy}{dx} + h_z \dfrac{dz}{dx} \end{cases} \Rightarrow \begin{cases} \dfrac{dy}{dx} = \dfrac{g_z h_x - g_x h_z}{g_y h_z - g_z h_y} \\ \dfrac{dz}{dx} = \dfrac{g_x h_y - g_y h_x}{g_y h_z - g_z h_y} \end{cases}$$

回代至第一个方程

$$\frac{du}{dx} = \frac{f_x(g_y h_z - g_z h_y) + f_y(g_z h_x - g_x h_z) + f_z(g_x h_y - g_y h_x)}{g_y h_z - g_z h_y}$$

$$= \frac{\begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix}}{\begin{vmatrix} g_y & g_z \\ h_y & h_z \end{vmatrix}}$$

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A-30. 使用数学归纳法证明：

① $n=1$ 时，目标：证明 $\left(\frac{\partial u}{\partial y}=\varphi(z)\frac{\partial u}{\partial x}\right)$

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial x}=1+y\varphi'(z)\frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y}=\varphi(z)+y\cdot\varphi'(z)\frac{\partial z}{\partial y} \end{cases}$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{1}{1-y\varphi'(z)} \quad \frac{\partial z}{\partial y}=\frac{\varphi(z)}{1-y\varphi'(z)} \quad \left(\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}\cdot\varphi(z)\right)$$

$$\Rightarrow \frac{\partial u}{\partial y}=\varphi(z)\frac{\partial u}{\partial x}.$$ 结论成立.

② 假设 $n=k$ 时. $\frac{\partial^k u}{\partial y^k}=\frac{\partial^{k-1}}{\partial x^{k-1}}\left[\varphi^k(z)\frac{\partial u}{\partial x}\right]$

$$\therefore \frac{\partial^{k+1}u}{\partial y^{k+1}}=\frac{\partial}{\partial y}\left(\frac{\partial^k u}{\partial y^k}\right)=\frac{\partial}{\partial y}\left[\frac{\partial^{k-1}}{\partial x^{k-1}}\left(\varphi^k(z)\frac{\partial u}{\partial x}\right)\right]$$

$$=\frac{\partial^{k-1}}{\partial x^{k-1}}\left[\frac{\partial}{\partial y}\left(\varphi^k(z)\frac{\partial u}{\partial x}\right)\right]$$

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\varphi(z)\frac{\partial u}{\partial x}\right)$

$$=\frac{\partial^{k-1}}{\partial x^{k-1}}\left[k\varphi^{k-1}(z)\varphi'(z)\frac{\partial z}{\partial y}\cdot\frac{\partial u}{\partial x}+\varphi^k(z)\frac{\partial^2 u}{\partial x\partial y}\right]$$

$$=\frac{\partial^{k-1}}{\partial x^{k-1}}\left[k\varphi^{k-1}(z)\varphi'(z)\varphi(z)\frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial x}+\varphi^k(z)\varphi'(z)\frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial x}+\varphi^k(z)\varphi(z)\frac{\partial^2 u}{\partial x^2}\right]$$

$$=\frac{\partial^{k-1}}{\partial x^{k-1}}\left[k\varphi^k(z)\varphi'(z)\frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial x}+\varphi^k(z)\varphi'(z)\frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial x}+\varphi^{k+1}(z)\frac{\partial^2 u}{\partial x^2}\right]$$

$$=\frac{\partial^{k-1}}{\partial x^{k-1}}\left[(k+1)\varphi^k\varphi^k(z)\varphi'(z)\frac{\partial z}{\partial x}\frac{\partial u}{\partial x}+\varphi^{k+1}(z)\frac{\partial^2 u}{\partial x^2}\right]$$

☆
$$=\frac{\partial^{k-1}}{\partial x^{k-1}}\frac{\partial}{\partial x}\left[\varphi^{k+1}(z)\frac{\partial u}{\partial x}\right]=\frac{\partial^k}{\partial x^k}\left[\varphi^{k+1}(z)\frac{\partial u}{\partial x}\right]$$

$\therefore n=k+1$ 时也成立

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31. $z_x = \alpha + \alpha_x \cdot x + \varphi'(\alpha)\cdot\alpha_x\cdot y + \psi'(\alpha)\cdot\alpha_x = \alpha + \alpha_x(\underbrace{x+\varphi(\alpha)y+\psi'(\alpha)}_{=0}) = \alpha$.

$z_y = \underline{\alpha_y\cdot x + \varphi'(\alpha)\cdot\alpha_y\cdot y} + \varphi(\alpha) + \underline{\psi'(\alpha)\alpha_y} = \varphi(\alpha) + \alpha_y(\underbrace{x+\varphi(\alpha)y+\psi(\alpha)}_{=0}) = \varphi(\alpha)$

$\therefore z_{xx} = \alpha_x \quad z_{xy} = \alpha_y \quad z_{yx} = \varphi'(\alpha)\cdot\alpha_x \quad z_{yy} = \varphi'(\alpha)\cdot\alpha_y$

$\therefore z_{xx}\cdot z_{yy} - (z_{xy})^2 = z_{xx}\cdot z_{yy} - z_{xy}\cdot z_{yx} = \varphi'(\alpha)\alpha_x\alpha_y - \varphi'(\alpha)\alpha_x\alpha_y = 0$.

32.

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial x} = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial y} + \dfrac{\partial u}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial y} = \lambda_1\dfrac{\partial u}{\partial \xi} + \lambda_2\dfrac{\partial u}{\partial \eta} \end{cases}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right] + \left[\frac{\partial^2 u}{\partial \eta\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\right]$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \quad ①$$

$$\Rightarrow \frac{\partial^2 u}{\partial x\partial y} = \left[\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right] + \left[\frac{\partial^2 u}{\partial \eta\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y}\right]$$

$$= \lambda_1\frac{\partial^2 u}{\partial \xi^2} + (\lambda_1+\lambda_2)\frac{\partial^2 u}{\partial \xi\partial \eta} + \lambda_2\frac{\partial^2 u}{\partial \eta^2} \quad ②$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \lambda_1\left[\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right] + \lambda_2\left[\frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y}\right]$$

$$= \lambda_1^2\frac{\partial^2 u}{\partial \xi^2} + 2\lambda_1\lambda_2\frac{\partial^2 u}{\partial \xi\partial \eta} + \lambda_2^2\frac{\partial^2 u}{\partial \eta^2} \quad ③$$

①+2②+③ 代入合并得

$$\underbrace{(A+2B\lambda_1+C\lambda_1^2)}_{\text{令之为0}}\frac{\partial^2 u}{\partial \xi^2} + (2A+2B(\lambda_1+\lambda_2)+2C\lambda_1\lambda_2)\frac{\partial^2 u}{\partial \xi\partial \eta} + \underbrace{(A+2B\lambda_2+C\lambda_2^2)}_{\text{令之为0}}\frac{\partial^2 u}{\partial \eta^2} = 0$$

$\because \lambda_1、\lambda_2$ 是 $Cx^2+2Bx+A=0$ 的根 $(B^2-AC>0)$

且 $\lambda_1 \neq \lambda_2$

(若 $\lambda_1=\lambda_2$，也为0，舍去)

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33.

(1) $\frac{\partial w}{\partial x}=\frac{1}{z}\cdot\frac{\partial z}{\partial x}-1 \quad \frac{\partial w}{\partial y}=\frac{1}{z}\cdot\frac{\partial z}{\partial y}-1 \quad \frac{\partial z}{\partial x}=z\cdot\frac{\partial w}{\partial x}+z \quad \frac{\partial z}{\partial y}=z\frac{\partial w}{\partial y}+z$

$\therefore yz\frac{\partial w}{\partial x}-xz\frac{\partial w}{\partial y}+(y-x)z=(y-x)z \qquad \therefore z\left(y\frac{\partial w}{\partial x}-x\frac{\partial w}{\partial y}\right)=0$

若 $z\neq0$ 有 $x\frac{\partial w}{\partial y}=y\frac{\partial w}{\partial x}$ （若 $z=0$ 也成立） 对于 $w=w(u,v)$

$\therefore \frac{\partial w}{\partial x}=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}=2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\cdot\frac{\partial w}{\partial v}$

$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}=2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}$

$\therefore 2xy\frac{\partial w}{\partial u}-\frac{x}{y^2}\frac{\partial w}{\partial v}=2xy\frac{\partial w}{\partial u}-\frac{y}{x^2}\frac{\partial w}{\partial v} \Rightarrow (x^3-y^3)\frac{\partial w}{\partial v}=0$

若 $y\neq x$ 有 $\frac{\partial w}{\partial v}=0$ （若 $y=x$ 也成立）

(2) $\frac{\partial w}{\partial y}=-1+x\cdot\frac{\partial z}{\partial y} \qquad \therefore \frac{\partial z}{\partial y}=\frac{1}{x}+\frac{\partial w}{x\partial y} \qquad \therefore \frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 w}{x\partial y^2}$

$\therefore \frac{y}{x}\frac{\partial^2 w}{\partial y^2}+\frac{2}{x}\left(\frac{\partial w}{\partial y}+1\right)=\frac{2}{x}$ 即 $y\frac{\partial^2 w}{\partial y^2}+2\frac{\partial w}{\partial y}=0$.

$y\neq0$ 时. $\frac{\partial w}{\partial y}=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}=-\frac{u}{y}\frac{\partial w}{\partial u}$

$\frac{\partial^2 w}{\partial y^2}=\frac{u}{y^2}\frac{\partial w}{\partial u}-\frac{1}{y}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial y}$

$=-\frac{u}{y}\left(\frac{\partial^2 w}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 w}{\partial v\partial u}\cdot\frac{\partial v}{\partial y}\right)=\frac{2u}{y^2}\cdot\frac{\partial w}{\partial u}+\frac{u^2}{y^2}\frac{\partial^2 w}{\partial u^2}$

$\Rightarrow \frac{2u}{y}\cdot\frac{\partial w}{\partial u}+\frac{u^2}{y}\cdot\frac{\partial^2 w}{\partial u^2}-\frac{2u}{y}\cdot\frac{\partial w}{\partial u}=0$

即 $\frac{u^2}{y}\cdot\frac{\partial^2 w}{\partial u^2}=0$ （$u\neq0$ 时 $\Rightarrow\frac{\partial^2 w}{\partial u^2}=0$） $\Rightarrow \frac{\partial^2 w}{\partial u^2}=0$

34.

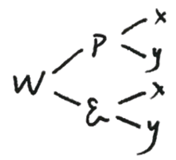
$\because p=z_x,\ q=z_y$

$\therefore W_p = x+px_p+qy_p-(z_x\cdot x_p+z_y\cdot y_p) = x+px_p+qy_p-px_p-qy_p = x$

$W_q = px_q+y+qy_q-(z_x\cdot x_q+z_y\cdot y_q) = y+px_q+qy_q-px_q-qy_q = y$

记 $W_p(p,q) = W_p(p(x,y),q(x,y))$

$\because W_p = x$



两边对 x 求导: $W_{pp}\cdot p_x + W_{pq}\cdot q_x = 1$

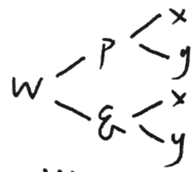
$\therefore W_{pp}\cdot z_{xx} + W_{pq}\cdot z_{yx} = W_{pp}z_{xx}+W_{pq}\cdot z_{xy} = 1 \cdots ①$

两边对 y 求导: $W_{pp}\cdot p_y + W_{pq}\cdot q_y = 0$

$\therefore W_{pp}\cdot z_{xy} + W_{pq}\cdot z_{yy} = 0 \quad \cdots ②$

记 $W_q(p,q) = W_q(p(x,y),q(x,y))$

$\because W_q = y$



两边对 x 求导: $W_{qp}\cdot p_x + W_{qq}\cdot q_x = 0$

$\therefore W_{qp}\cdot z_{xx} + W_{qq}\cdot z_{yx} = W_{pq}\cdot z_{xx}+W_{qq}\cdot z_{xy} = 0 \cdots ③$

两边对 y 求导: $W_{qp}\cdot p_y + W_{qq}\cdot q_y = 1$

$\therefore W_{qp}\cdot z_{xy}+W_{qq}\cdot z_{yy} = W_{pq}\cdot z_{xy}+W_{qq}\cdot z_{yy} = 1 \cdots ④$

$$①②③④ \Leftrightarrow \begin{pmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy}\end{pmatrix}\begin{pmatrix} W_{pp} & W_{pq} \\ W_{pq} & W_{qq}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$$

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35. $P: x+y+xy=1$ $(1+x)y=1-x$ $y=\frac{1+x}{1-x}$. $y_x=\frac{-2}{(1-x)^2}=-\frac{1+y}{1+x}$

切线方程为 $y-y_0=-\frac{1+y_0}{1+x_0}(x-x_0)$

即 $y=-\frac{1+y_0}{1+x_0}(x-x_0)+y_0$

36. $x_t=1$ $y_t=2t$ $z_t=3t^2$ 记 $\vec{l}=(1,2t,3t^2)$

记曲线上任一点 (t_0,t_0^2,t_0^3)

切线方程为 $\frac{x-t_0}{1}=\frac{y-t_0^2}{2t_0}=\frac{z-t_0^3}{3t_0^2}$

平面 $x+2y+z=4$ 的法向 $\vec{n}=(1,2,1)$

令 $\vec{n}\cdot\vec{l}|_{t_0}=1+4t_0+3t_0^2=0$ $\therefore t_0=-1$ 或 $-\frac{1}{3}$

$\therefore (-1,1,-1)$ 与 $(-\frac{1}{3},\frac{1}{9},-\frac{1}{27})$ 两点符合条件

37. 记 $F(x,y,z)=xyz-a^3=0$ $\frac{\partial F}{\partial x}=yz$ $\frac{\partial F}{\partial y}=xz$ $\frac{\partial F}{\partial z}=xy$

曲面上任一点 $P_0(x_0,y_0,z_0)$ 的切向为 (y_0z_0,x_0z_0,x_0y_0).

切平面为 $y_0z_0(x-x_0)+x_0z_0(y-y_0)+x_0y_0(z-z_0)=0$

$\therefore$ 四面体四顶点分别为 $(0,0,0)$, $(0,0,3z_0)$, $(0,3y_0,0)$, $(3x_0,0,0)$

$V=\frac{1}{3}\times\frac{1}{2}|3x_0\cdot 3y_0|\times|3z_0|=\frac{9}{2}|x_0y_0z_0|=\frac{9}{2}a^3$ 为常数.

得证.

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38. 记 $f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}=0$

$f_x=\frac{1}{2\sqrt{x}}$ $f_y=\frac{1}{2\sqrt{y}}$ $f_z=\frac{1}{2\sqrt{z}}$

曲面上任一点 (x_0,y_0,z_0) 的切平面为 $\frac{x-x_0}{\sqrt{x_0}}+\frac{y-y_0}{\sqrt{y_0}}+\frac{z-z_0}{\sqrt{z_0}}=0$

$\therefore \frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}$

$\therefore \frac{x}{\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})}+\frac{y}{\sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})}+\frac{z}{\sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})}=1$

$\therefore$ 曲面在x、y、z轴上截距分别为 $\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})$、$\sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+z_0)$、$\sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})$

$\therefore$ 截距和为 $(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})^2=a$

39. 记 $F=x^2+y^2-a^2$ $G=z-xy$

$\vec{l_F}=(F_x,F_y,F_z)=2(x,y,0)$ $\vec{l_G}=(-y,-x,1)$

对F与G任一交点 $P_0(x_0,y_0,z_0)$，$\left(\begin{cases}x_0^2+y_0^2=a^2\\ z_0=x_0y_0\end{cases}\right)$

$$\cos\langle\vec{l_F},\vec{l_G}\rangle=\frac{\vec{l_F}\cdot\vec{l_G}}{|\vec{l_F}||\vec{l_G}|}=\frac{-2x_0y_0}{\sqrt{x_0^2+y_0^2}\cdot\sqrt{x_0^2+y_0^2+1}}=\frac{-2z_0}{a\sqrt{a^2+1}}$$

$\therefore$ 在 $P_0(x_0,y_0,z_0)$ 处夹角为 $\arccos\left(-\frac{2z_0}{a\sqrt{a^2+1}}\right)$ 即 $\arccos\frac{2z_0}{a\sqrt{a^2+1}}$

40. 证：记 $F(x,y)=f(x,y)-g(x,y)$

$\because$ 对 $\forall(x,y)\in\partial D$ 有 $f=g$. 即：在闭域 D 的边界上，$F=0$

在 D 的内部即 D°：

① 若对 $\forall(x_0,y_0)\in D^\circ$ 有 $F(x_0,y_0)=0$. 显然结论成立

② 若 $\exists(x_0,y_0)\in D^\circ$ 使 $F(x_0,y_0)\neq0$.

1° 若 $F(x_0,y_0)>0$ 则 D° 中能取到 F 的最大值

$\therefore D^\circ$ 中能取到 F 的极大值 $\therefore F_x=F_y=0$ 即 $\nabla F(x_0)=0$

$\therefore \nabla f(x_0)=\nabla g(x_0)$

2° 若 $F(x_0,y_0)<0$ 则 D° 中能取到 F 的最小值

$\therefore D^\circ$ 中能取到 F 的极小值 $\therefore F_x=F_y=0$ 即 $\nabla F(x_0)=0$

$\therefore \nabla f(x_0)=\nabla g(x_0)$

41. [引理]. f 是凸函数 $\Rightarrow$ 对 $\forall X_1、X_2\in D$，有 $f(x_2)-f(x_1)\geq \mathrm{grad}\, f(x_1)\cdot\frac{X_2-X_1}{\|X_2-X_1\|}=\langle\nabla f(X_1),\frac{X_2-X_1}{\|X_2-X_1\|}\rangle$
数量积

(证明：$\Rightarrow$：f 是 D 上凸函数 $\therefore \forall x_1、x_2\in D$ 及 $t\in(0,1)$ 有 $f(tx_2+(1-t)x_1)\leq tf(x_2)+(1-t)f(x_1)$

即 $\frac{f(tX_2+(1-t)X_1)-f(X_1)}{t}\leq f(X_2)-f(X_1)$ 令 $t\to0$. 左式 $=\frac{\partial f}{\partial t}\Big|_{X_1}=\nabla f(X_1)\cdot\frac{X_2-X_1}{\|X_2-X_1\|}$)

(其逆命题也成立，可自行证明)

证：$\because X_0$ 是 $f(X)$ 临界点 $\therefore \nabla f(X_0)=0$.

$\therefore$ 对 $\forall X\in\mathbb{R}^n$. $\because$ f 是凸函数.

由引理 $f(X)-f(X_0)\geq\langle\nabla f(X_0),\frac{X-X_0}{\|X-X_0\|}\rangle=\langle 0,\frac{X-X_0}{\|X-X_0\|}\rangle=0$

对 $\forall X\in\mathbb{R}^n$ 有 $f(X)\geq f(X_0)$ $\therefore X_0$ 是 $f(X)$ 最小值点

42. 在(x_0,y_0)点对函数作Taylor展开. 又对$\forall(x,y)$

$$f(x,y)-f(x_0,y_0)=\underbrace{\nabla f(x_0,y_0)}_{\text{取极值}\ =0}+\frac{1}{2}(x-x_0,y-y_0)\cdot H_f(x_0,y_0)\begin{pmatrix}x-x_0\\y-y_0\end{pmatrix}$$

(f是二元函数 ⇒ 三阶及以上偏导均为0).

不妨设f取到极大值于X_0点，则$H_f\leq 0$ $\therefore\forall\ f(x,y)-f(x_0,y_0)\leq 0$

$\therefore$ f在(x_0,y_0)处取得max.

43. $f_x=-(1+e^y)\sin x=0$ $f_y=\cos x e^y-(y+1)e^y$ 当$\nabla f=0$

令$\begin{cases}(1+e^y)\sin x=0\\ \cos x e^y=(y+1)e^y\end{cases}\Rightarrow\begin{cases}x_1=2k\pi\\y_1=0\end{cases}$ 或 $\begin{cases}x_2=(2k+1)\pi\\y_2=-2\end{cases}$

$\because f_{xx}=-(1+e^y)\cos x$ $f_{xy}=-\sin x e^y$ $f_{yy}=\cos x e^y-(y+2)e^y$

1) 在$(2k\pi,0)$点$f_{xx}\cdot f_{yy}-f_{xy}^2=(-2)\cdot(-1)-0=2>0$，$f_{xx}=-2<0$

$\therefore H_f(X_1)<0$ $\therefore f$在$\forall(2k\pi,0)$ $(k\in\mathbb{Z})$上取得极大值.

2) 在$((2k+1)\pi,-2)$点$f_{xx}\cdot f_{yy}-f_{xy}^2=(1+\frac{1}{e^2})\cdot\frac{-1}{e^2}<0$ 不取极值

$\therefore f$有无穷多个极大值但无极小值.

44. 若$f(x,y)$在$\mathbb{R}^2$上存在点(x_0,y_0)使$f(x_0,y_0)$为f极大值

$\therefore(x_0,y_0)$能使：$\begin{cases}f_{xx}f_{yy}-f_{xy}^2\geq 0\\ f_{xx}\leq 0\end{cases}$

$\because$ 由题可知$f_{xx}+f_{yy}>0$ 又$f_{xx}\leq 0$ $\therefore f_{yy}\geq 0$ $\therefore f_{xx}\cdot f_{yy}\leq 0$

又$\because f_{xy}^2\geq 0$. $\because f_{xx}f_{yy}-f_{xy}^2\geq 0$ $\therefore f_{xx}f_{yy}=f_{xy}=0$

此时f在(x_0,y_0)显然不取得极值，矛盾

$\therefore f$无极大值点

45.

STEP1 令 $\begin{cases} \sigma_a = \sum 2x_i(ax_i+b-y_i)=0 \\ \sigma_b = -\sum 2(ax_i+b-y_i)=0 \end{cases} \Rightarrow \begin{cases} \left(\sum\limits_{i=1}^n x_i^2\right)a + \left(\sum\limits_{i=1}^n x_i\right)b = \sum\limits_{i=1}^n x_i y_i \\ \left(\sum\limits_{i=1}^n x_i\right)a + nb = \sum\limits_{i=1}^n y_i \end{cases}$

$\because \begin{pmatrix} \sum\limits_{i=1}^n x_i^2 & \sum\limits_{i=1}^n x_i \\ \sum\limits_{i=1}^n x_i & n \end{pmatrix}$ 是实对称矩阵，故其可逆

$\because n\sum\limits_{i=1}^n x_i^2 > \left(\sum\limits_{i=1}^n x_i\right)^2$ （由cauchy-schwarz不等式）

$\sum\limits_{i=1}^n x_i = \left(\sum\limits_{i=1}^n 1\cdot x_i\right) < \left(\sum\limits_{i=1}^n 1^2\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^n x_i^2\right)^{\frac{1}{2}}$ （由hölder不等式）

$\therefore \det\begin{pmatrix} \sum\limits_{i=1}^n x_i^2 & \sum\limits_{i=1}^n x_i \\ \sum\limits_{i=1}^n x_i & n \end{pmatrix} > 0$ 又 $\sum\limits_{i=1}^n x_i^2 > 0$ 故此矩阵正定

$\therefore$ 临界点 $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^n x_i^2 & \sum\limits_{i=1}^n x_i \\ \sum\limits_{i=1}^n x_i & n \end{pmatrix}^{-1} \begin{pmatrix} \sum\limits_{i=1}^n x_i y_i \\ \sum\limits_{i=1}^n y_i \end{pmatrix}$

STEP 2 下证 σ 在 $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$ 取到min

① 当 $\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| \to +\infty$ 时，$\sigma(a,b) \to +\infty$

$\exists L>0 \Rightarrow \sigma(a,b)$ 在 $\left\{\begin{pmatrix} a \\ b \end{pmatrix} : \left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| \le L\right\}$ 上取到min

② $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$ 是 σ 的唯一临界点

由①②，σ 在 $\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$ 取得min

记 $D = \begin{vmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{vmatrix}$ $D_1 = \begin{vmatrix} \sum x_i y_i & \sum x_i \\ \sum y_i & n \end{vmatrix}$ $D_2 = \begin{vmatrix} \sum x_i^2 & \sum x_i y_i \\ \sum x_i & \sum y_i \end{vmatrix}$

当 $D \neq 0$ 时. $a_0 = \frac{D_1}{D}$，$b_0 = \frac{D_2}{D}$

46. 记 $h(x)=\int \frac{1}{(1+x^2)^2}dx$

$$\because \int (ax+b-\frac{1}{1+x^2})^2 dx = \int \left(a^2x^2+b^2+\frac{1}{(1+x^2)^2}+2abx-\frac{2b}{1+x^2}-\frac{2ax}{1+x^2}\right)dx$$

$$=\frac{a^2}{3}x^3+b^2x+abx^2-2b\arctan x-a\ln(1+x^2)+h(x)+C$$

$$\therefore \int_0^1 (ax+b-\frac{1}{1+x^2})^2 dx = \frac{a^2}{3}+b^2+ab-\frac{\pi}{2}b-a\ln 2+h(1)-h(0) \triangleq g(a,b)$$

令 $\begin{cases} g_a=\frac{2}{3}a+b-\ln 2=0 \\ g_b=2b-\frac{\pi}{2}+a=0 \end{cases} \Rightarrow \begin{cases} a=6\ln 2-\frac{3}{2}\pi \\ b=\pi-3\ln 2 \end{cases}$

经检验，f 在 $(6\ln 2-\frac{3}{2}\pi,\ \pi-3\ln 2)$ 取得极小值，即取得最小值.

47. 记 $f(a,b,c)=\sum_{i=1}^{n}(ax_i^2+bx_i+c-y_i)^2$

$\therefore f_a=2\sum_{i=1}^{n}x_i^2(ax_i^2+bx_i+c-y_i)$ $\quad f_b=2\sum_{i=1}^{n}x_i(ax_i^2+bx_i+c-y_i)$ $\quad f_c=2\sum_{i=1}^{n}(ax_i^2+bx_i+c-y_i)$

令 $f_a=f_b=f_c=0 \Rightarrow \begin{cases} a\sum x_i^4+b\sum x_i^3+c\sum x_i^2=\sum x_i^2y_i \\ a\sum x_i^3+b\sum x_i^2+c\sum x_i=\sum x_iy_i \\ a\sum x_i^2+b\sum x_i+cn=\sum y_i \end{cases}$

$$D=\begin{vmatrix} \sum x_i^4 & \sum x_i^3 & \sum x_i^2 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i \\ \sum x_i^2 & \sum x_i & n \end{vmatrix} \quad D_1=\begin{vmatrix} \sum x_i^2y_i & \sum x_i^3 & \sum x_i^2 \\ \sum x_iy_i & \sum x_i^2 & \sum x_i \\ \sum y_i & \sum x_i & n \end{vmatrix} \quad D_2=\begin{vmatrix} \sum x_i^4 & \sum x_i^2y_i & \sum x_i^2 \\ \sum x_i^3 & \sum x_iy_i & \sum x_i \\ \sum x_i^2 & \sum y_i & n \end{vmatrix}$$

$$D_3=\begin{vmatrix} \sum x_i^4 & \sum x_i^3 & \sum x_i^2y_i \\ \sum x_i^3 & \sum x_i^2 & \sum x_iy_i \\ \sum x_i^2 & \sum x_i & \sum y_i \end{vmatrix}$$

当 $D\neq 0$ 时. $a=\frac{D_1}{D}$ $\quad b=\frac{D_2}{D}$ $\quad c=\frac{D_3}{D}$

经检验. f 在 $(\frac{D_1}{D},\frac{D_2}{D},\frac{D_3}{D})$ 取到min .

48、

记 $F(x,y)=x^3+y^3-3xy$

$\therefore F_x=3x^2-3y \quad F_y=3y^2-3x$

$\because$ F、F_x、F_y 都是关于x、y、z的初等函数 且 $x\neq y^2$

$\therefore \begin{cases} ① F、F_x、F_y 在 B(x_0,y_0) 连续 \\ ② F(x_0,y_0)=0 \\ ③ F_y(x_0,y_0)\neq 0 \end{cases}$

由隐函数存在定理，$F(x)=0$ 可唯一确定函数 $y=y(x)$.

且 $y'(x)=-\dfrac{F_x}{F_y}=\dfrac{y-x^2}{y^2-x}$

令 $y'(x)=0$ $\quad \therefore \begin{cases} y-x^2=0 \\ x^3+y^3-3xy=0 \end{cases} \Rightarrow \begin{cases} x=\sqrt[3]{2} \\ y=\sqrt[3]{4} \end{cases}$

又 $y''(x)=\dfrac{(y'(x)-2x)(y^2-x)-(y-x^2)(2y\cdot y'(x)-1)}{(y^2-x)^2}=\dfrac{-2\sqrt[3]{2}}{\sqrt[3]{16}-\sqrt[3]{2}}<0$

$\therefore y(x)$ 在 $x=\sqrt[3]{2}$ 处取得极大值 $\sqrt[3]{4}$.

49. 记 $L(x_1,\cdots,x_n)=\sum x_i^2+\lambda(\sum a_i x_i-1)$

令 $\begin{cases} 2x_i+\lambda a_i=0 \quad (i=1,\cdots,n) \\ \sum\limits_{i=1}^{n} a_i x_i=1 \end{cases} \Rightarrow \lambda=\dfrac{-2}{\sum a_i^2}$ $\quad \because H_f=\begin{pmatrix} 2 & & \\ & 2 & \\ & & \ddots \\ & & & 2\end{pmatrix}>0$

$\therefore x_i=\dfrac{a_i}{\sum a_i^2}$ $\quad \therefore \left(\dfrac{a_1}{\sum a_i^2},\dfrac{a_2}{\sum a_i^2},\cdots\dfrac{a_n}{\sum a_i^2}\right)$ 是其极小值点

$\therefore f_{极小}=f\left(\dfrac{a_1}{\sum a_i^2},\cdots,\dfrac{a_n}{\sum a_i^2}\right)=\dfrac{\sum a_i^2}{(\sum a_i^2)^2}=\dfrac{1}{\sum a_i^2}$

50. 记 $L(x_1,\cdots,x_n)=\prod x_i^2+\lambda(\sum x_i^2-r^2)$

令 $\begin{cases} 2x_i\cdot\dfrac{\prod x_i^2}{x_i^2}+2\lambda x_i=0 \quad (i=1,\cdots,n) \\ \sum x_i^2=r^2 \end{cases} \Rightarrow \prod x_i^2+\lambda x_i^2=0 \xRightarrow{n式相加} n\cdot\prod x_i^2+\lambda(\sum x_i^2)=0$

$\therefore n\prod x_i^2+\lambda r^2=0$

又 $\lambda=-\dfrac{\pi x_i^2}{x_i^2}$ $\quad \therefore x_i^2=\dfrac{r^2}{n}$ $\quad (i=1,2,\cdots,n)$ $\quad \therefore u_{max}=u\left(\dfrac{r}{\sqrt n},\cdots,\dfrac{r}{\sqrt n}\right)=\dfrac{r^{2n}}{n^n}$

51. 令 $d=x^2+y^2+z^2$ $\quad$ 令 $L(x,y,z)=x^2+y^2+z^2+\lambda(a_1x+b_1y+c_1z-1)+\mu(a_2x+b_2y+c_2z-1)$

令 $\begin{cases} 2x+\lambda a_1+\mu a_2=0 & ① \\ 2y+\lambda b_1+\mu b_2=0 & ② \\ 2z+\lambda c_1+\mu c_1=0 & ③ \\ a_1x+b_1y+c_1z=1 & ④ \\ a_2x+b_2y+c_2z=1 & ⑤ \end{cases}$ 设其解为 (x_0,y_0) $\quad H_f=\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2\end{pmatrix}>0$ $\quad \therefore f(x_0,y_0)=f_{min}$

$\therefore d_m=\sqrt{x_0^2+y_0^2+z_0^2}=\sqrt{-\frac{1}{2}(\lambda+\mu)}=\sqrt{\dfrac{a_1^2+b_1^2+c_1^2+a_2^2+b_2^2+c_2^2-2(a_1a_2+b_1b_2+c_1c_2)}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2}}$

52. 令 $f(x,y,z)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1$ $\quad \nabla f(x,y,z)=\left(\dfrac{2x}{a^2},\dfrac{2y}{b^2},\dfrac{2z}{c^2}\right)$ $\quad \forall(x_0,y_0,z_0)\in$ Ⅰ卦限 $(x_0y_0z_0\neq0)$

切平面方程为：$\dfrac{2x_0}{a^2}(x-x_0)+\dfrac{2y_0}{b^2}(y-y_0)+\dfrac{2z_0}{c^2}(z-z_0)=0$

分别令 x,y,z 中二者为零 得其与坐标轴交点 $\left(0,0,\dfrac{c^2}{z_0}\right)$ $\left(0,\dfrac{b^2}{y_0},0\right)$ $\left(\dfrac{a^2}{x_0},0,0\right)$

$\therefore$ 体积 $V=\dfrac{1}{3}\times\left(\dfrac{1}{2}\cdot\dfrac{a^2b^2}{x_0y_0}\right)\cdot\dfrac{c^2}{z_0}=\dfrac{a^2b^2c^2}{6}\cdot\dfrac{1}{x_0y_0z_0}$

$\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\geqslant 3\sqrt[3]{\dfrac{x^2y^2z^2}{a^2b^2c^2}}\right.$ $\quad \left.\therefore xyz\leqslant\dfrac{abc}{3\sqrt3}\right)$

$\therefore V\geqslant\dfrac{a^2b^2c^2}{6}\cdot\dfrac{3\sqrt3}{abc}=\dfrac{\sqrt3}{2}abc$