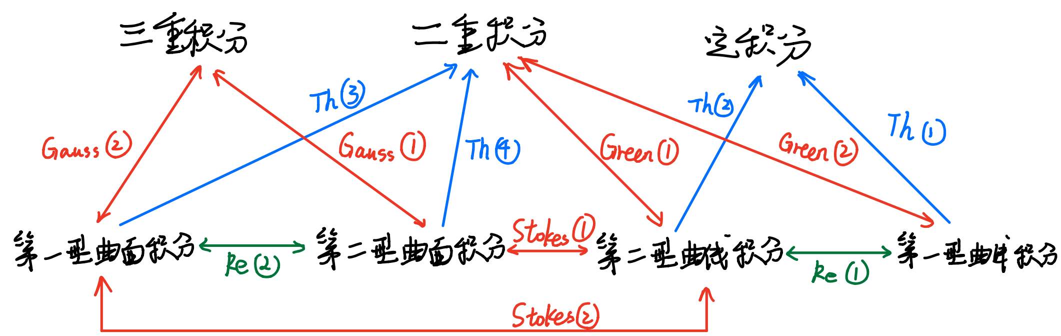
第十三章 曲线积分与曲面积分



Th ① $\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$

$$\int_L f(x,y,z)\,ds=\int_a^b f(x(t),y(t),z(t))\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\,dt$$

Th ② $\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$

$$\int_L P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz$$

$$=\int_a^b \Big[P(x(t),y(t),z(t))\cdot x'(t)+Q\cdot y'(t)+R\cdot z'(t)\Big]\,dt$$

Re ① $\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t) \end{cases}$

$$\vec{\tau}=(\cos\alpha,\cos\beta,\cos\gamma)=\frac{1}{\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}}\big(x'(t),y'(t),z'(t)\big)$$ (单位切向量)

$$\int_L P\,dx+Q\,dy+R\,dz=\int_L (P\cos\alpha+Q\cos\beta+R\cos\gamma)\,ds$$

Th ③ (1) $S: z=z(x,y)$

$$\iint_S f(x,y,z)\,dS=\iint_{D_{xy}} f(x,y,z(x,y))\sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

(2) $S: \begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$

$$\iint_S f(x,y,z)\,dS=\iint_D f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}\,du\,dv$$

其中：$E=x_u^2+y_u^2+z_u^2$, $G=x_v^2+y_v^2+z_v^2$, $F=x_u\cdot x_v+y_u\cdot y_v+z_u\cdot z_v$

Th ④ (1) $S: z=z(x,y)$

$$\iint_S P\,dy\,dz+Q\,dz\,dx+R\,dx\,dy$$

$$=\pm\iint_{D_{xy}}\Big[-P(x,y,z(x,y))\frac{\partial z}{\partial x}-Q(x,y,z(x,y))\frac{\partial z}{\partial y}+R(x,y,z(x,y))\Big]\,dx\,dy$$

(S上侧为正侧时，取正号)

(2) $S: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ $\quad A = \dfrac{D(y,z)}{D(u,v)} \quad B = \dfrac{D(z,x)}{D(u,v)} \quad C = \dfrac{D(x,y)}{D(u,v)}$

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$
$$= \pm\iint_D P(x(u,v),y(u,v),z(u,v))A + Q(x(u,v),y(u,v),z(u,v))B + R(x(u,v),y(u,v),z(u,v))C \; du\,dv$$

Re ②

(1) $S: \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ $\quad \vec{n} = (\cos\alpha, \cos\beta, \cos\gamma) = \dfrac{1}{\sqrt{A^2+B^2+C^2}}(A, B, C)$

$\vec{\tau}$：单位外法向量

(2) $S: F(x,y,z) = 0 \quad \vec{n} = (\cos\alpha, \cos\beta, \cos\gamma) = \dfrac{1}{\sqrt{F_x^2+F_y^2+F_z^2}}\left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}\right)$

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS$$

($dy\,dz = \cos\alpha\,dS$, $dz\,dx = \cos\beta\,dS$, $dx\,dy = \cos\gamma\,dS$，三者可互化)

Green ① $\quad \displaystyle\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Green ② $\quad \displaystyle\int_{\partial D} (P\cos i + Q\cos j)\,ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy$

$\vec{n} = (\cos i, \cos j)$ 是平面D的单位外法向量

Gauss ① $\quad \displaystyle\iint_{\partial\Omega} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$

Gauss ② $\quad \displaystyle\iint_{\partial\Omega} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$

$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ 是曲面$\partial\Omega$的单位外法向量

Stokes ① $\quad \displaystyle\int_{\partial S} P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Stokes ② $\quad \displaystyle\int_{\partial S} P\,dx + Q\,dy + R\,dz = \iint_S \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\right] ds$

$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ 是曲面S的单位外法向量

练习13.1

1. (1) 对于原方程作极坐标变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

可化为：$r^2 = a^2\cos 2\theta$ θ定义域：$[-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3}{4}\pi, \frac{5}{4}\pi]$

故双纽线关于θ的参数方程为 $\begin{cases} x = a\cos\theta\sqrt{\cos 2\theta} \\ y = a\sin\theta\sqrt{\cos 2\theta} \end{cases}$

$$x'(\theta) = \frac{a\sin\theta(4\cos^2\theta - 1)}{\sqrt{\cos 2\theta}} \qquad y'(\theta) = \frac{a\cos\theta(1 - 4\sin^2\theta)}{\sqrt{\cos 2\theta}}$$

$$\sqrt{x'^2 + y'^2} = \frac{a}{\sqrt{\cos 2\theta}}$$

$$\therefore \int_L |y|\,ds = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a\sin\theta\sqrt{\cos 2\theta} \cdot \frac{a}{\sqrt{\cos 2\theta}}\,d\theta + \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} a\sin\theta\sqrt{\cos 2\theta} \cdot \frac{a}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 4a^2\int_0^{\frac{\pi}{4}} \sin\theta\,d\theta = (4 - 2\sqrt{2})a^2$$

(2) $x'(t) = a(1 - \cos t)$ $\qquad y'(t) = a\sin t$

$$\sqrt{x'^2 + y'^2} = a\sqrt{2 - 2\cos t}$$

$$\therefore \int_L y^2\,ds = \int_0^{2\pi} a^2(1 - \cos t)^2 \cdot a\sqrt{2 - 2\cos t}\,dt$$

$$= \sqrt{2}a^3\int_0^{2\pi} (1 - \cos t)^{\frac{5}{2}}\,dt = \sqrt{2}a^3 \cdot \int_0^{2\pi} (\sqrt{2}\sin\frac{t}{2})^5\,dt$$

$$= 8a^3\int_0^{2\pi} \sin^5\frac{t}{2}\,dt = -16a^3\int_0^{2\pi} (1 - \cos^2\frac{t}{2})^2\,d\cos\frac{t}{2}$$

$$= 16a^3\int_0^1 (x^4 - 2x^2 + 1)\,dx = 16a^3(\frac{1}{5} - \frac{2}{3} + 1) = \frac{128}{15}a^3$$

2.

$$m = \int_L |y|\,ds \qquad 其中\ L: \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ z=0 \end{cases}$$

$$令 \begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases} \qquad \sqrt{x'^2+y'^2} = \sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$m = 2\int_0^{\pi} b\sin\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta$$

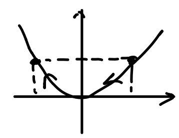
$$订子 \int b\sin\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta = -b\int\sqrt{a^2+(b^2-a^2)\cos^2\theta}\,d\cos\theta$$

$$= \begin{cases} -b\sqrt{b^2-a^2}\left[\frac{\cos\theta}{2}\sqrt{\frac{a^2}{b^2-a^2}+\cos^2\theta} + \frac{a^2}{2(b^2-a^2)}\ln\left|\cos\theta+\sqrt{\frac{a^2}{b^2-a^2}+\cos^2\theta}\right|\right]+C & a<b \\ -b\sqrt{a^2-b^2}\left[\frac{\cos\theta}{2}\sqrt{\frac{a^2}{a^2-b^2}+\cos^2\theta} + \frac{a^2}{2(a^2-b^2)}\arcsin\frac{ax}{\sqrt{a^2-b^2}}\right]+C & a>b \\ -ab & a=b \end{cases}$$

$$\therefore m = \begin{cases} 2b^2+\frac{2a^2b}{\sqrt{b^2-a^2}}\ln\frac{\sqrt{b^2-a^2}+b}{a} & a<b \\ 2b^2+\frac{2a^2b}{\sqrt{a^2-b^2}}\arcsin\frac{\sqrt{a^2-b^2}}{a} & a>b \\ 4ab & a=b \end{cases}$$

练习13.2

1. (1) $\begin{cases} x=x \\ y=x^2 \end{cases}$

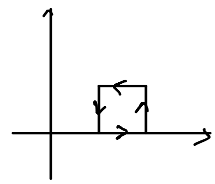


原式 $= \int_1^{-1} \left[(x^2-2x^3)+2x(x^4-2x^3)\right]dx$

$= \frac{x^6}{3}-\frac{4}{5}x^5-\frac{x^4}{2}+\frac{x^3}{3}\Big|_1^{-1} = \frac{14}{15}$

注：可以利用对称性避免一些项的计算.

(2)



原式 $= \int_1^2 x^2dx+\int_0^1(4-y^2)dy+\int_1^2(x^2+1)dx+\int_1^0(1-y^2)dy$

$= \int_0^1 3dy - \int_1^2 1dx = 1$

2. 证: 令 $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ 设 $t\in[a,b]$，t由 $a\to b$ 的过程即为L的方向.

$\therefore \int_L f(x^2+y^2)(xdx+ydy) = \int_L f(\varphi^2+\psi^2)(\varphi\cdot\varphi'+\psi\cdot\psi')dt$

$= \frac{1}{2}\int_L f(\varphi^2+\psi^2)\,d(\varphi^2+\psi^2)$

$= \frac{1}{2}\int_{\varphi^2(a)+\psi^2(a)}^{\varphi^2(b)+\psi^2(b)} f(r^2)\,dr^2$

其中 $r^2=x^2+y^2$ 由于L是光滑闭曲线，起点与终点重合

$\therefore \varphi^2(a)+\psi^2(a)=\varphi^2(b)+\psi^2(b)$

$\therefore$ 上式 $=0$

练习13.3

1. $z=\sqrt{4-x^2-y^2}$ $\quad\begin{cases} z_x=\dfrac{-x}{\sqrt{4-x^2-y^2}} \\ z_y=\dfrac{-y}{\sqrt{4-x^2-y^2}} \end{cases}$

$D:\begin{cases} x^2+y^2\le 3 \\ z=0 \end{cases}$

(1) 原式 $=\iint_D (x^2+y^2)\sqrt{4-x^2-y^2}\cdot\sqrt{1+\dfrac{x^2+y^2}{4-x^2-y^2}}\,dxdy$

$=\iint_D (x^2+y^2)\cdot\sqrt{4-x^2-y^2}\cdot\dfrac{2}{\sqrt{4-x^2-y^2}}\,dxdy$

$=2\iint_D (x^2+y^2)\,dxdy = 2\iint_{D'} r^3\,drd\theta$

$=2\int_0^{2\pi}d\theta\int_0^{\sqrt{3}} r^3\,dr = 9\pi$

(2) 原式 $=\iint_D \left(8-\sqrt{4-x^2-y^2}\right)\dfrac{2}{\sqrt{4-x^2-y^2}}\,dxdy$

$=\iint_D\left(\dfrac{16}{\sqrt{4-x^2-y^2}}-8\right)dxdy$

$=16\iint_D\dfrac{dxdy}{\sqrt{4-x^2-y^2}}-24\pi = 16\iint_{D'}\dfrac{r\,drd\theta}{\sqrt{4-r^2}}-24\pi$

$=-8\int_0^{2\pi}d\theta\int_4^1\dfrac{d(4-r^2)}{\sqrt{4-r^2}}-24\pi = 32\pi-24\pi=8\pi$

2、由几何体对称性，其质心必在 z 轴上. 设几何体区域为 V

以下由质心公式： $z_C = \dfrac{\int_m z\,dm}{\int_m dm}$

几何体在 xoy 平面投影为 D: $x^2+y^2 \le a^2\sin^2\beta$

$$\therefore \iint_V z\,dv = \iint_D z\cdot\sqrt{1+z_x^2+z_y^2}\,dxdy$$

$$= \iint_D \sqrt{a^2-x^2-y^2}\cdot\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,dxdy$$

$$= \iint_D a\,dxdy = a\cdot\pi a^2\sin^2\beta = \pi a^3\sin^2\beta$$

$$\iint_V dv = \iint_D \sqrt{1+z_x^2+z_y^2}\,dxdy = a\iint_D \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$$

$$= a\iint_{D'} \frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}} = \frac{a}{2}\int_0^{2\pi} d\theta \int_0^{a^2\sin^2\beta} \frac{dr^2}{\sqrt{a^2-r^2}}$$

$$= 2\pi a^2(1-\cos\beta)$$

$$\therefore \quad z_C = \frac{\pi a^3\sin^2\beta}{2\pi a^2(1-\cos\beta)} = \frac{a(1+\cos\beta)}{2}$$

$\therefore$ 质心坐标为 $\left(0, 0, \dfrac{a(1+\cos\beta)}{2}\right)$

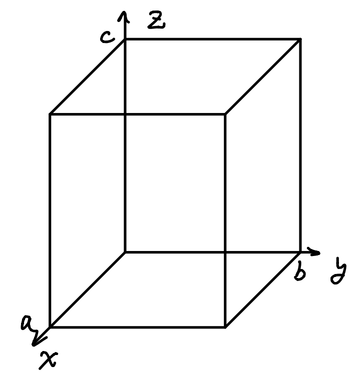
练习13.4

1.

(1) 解：

如图，分别记上、下、左、右、前、后六面为：

$S_1, S_2, S_3, S_4, S_5, S_6$



$$\therefore \iint_S f(x)dydz+g(y)dzdx+h(z)dxdy$$

$$=\iint_{S_1} h(z)dxdy+\iint_{S_2} h(z)dxdy+\iint_{S_3} g(y)dzdx+\iint_{S_4} g(y)dzdx+\iint_{S_5} f(x)dydz+\iint_{S_6} f(x)dydz$$

$$=-abh(c)-abh(0)-acg(0)-acg(b)-bcf(a)-bcf(0)$$

(2) 解： 令 $\begin{cases} x=a\sin\varphi\cos\theta \\ y=b\sin\varphi\sin\theta \\ z=c\cos\varphi \end{cases}$

$x_\varphi=a\cos\varphi\cos\theta \qquad y_\varphi=b\sin\theta\cos\varphi \qquad z_\varphi=-c\sin\varphi$

$x_\theta=-a\sin\varphi\sin\theta \qquad y_\theta=b\sin\varphi\cos\theta \qquad z_\theta=0$

$A=\begin{vmatrix} y_\varphi & z_\varphi \\ y_\theta & z_\theta \end{vmatrix}=bc\sin^2\varphi\cos\theta \qquad B=\begin{vmatrix} z_\varphi & x_\varphi \\ z_\theta & x_\theta \end{vmatrix}=ac\sin^2\varphi\sin\theta \qquad C=\begin{vmatrix} x_\varphi & y_\varphi \\ x_\theta & y_\theta \end{vmatrix}=ab\sin\varphi\cos\varphi$

$$\therefore \iint_S xdydz+ydzdx+zdxdy$$

$$=abc\iint_D(\sin^3\varphi\cos^2\theta+\sin^3\varphi\sin^2\theta+\sin\varphi\cos^2\varphi)\,d\theta d\varphi$$

$$=abc\iint_D \sin\varphi\,d\theta d\varphi=abc\int_0^\pi\sin\varphi\,d\varphi\int_0^{2\pi}d\theta=4abc\pi$$

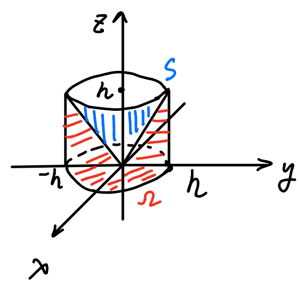
练习13.5

1、令 $Q(x,y)=x$，$P(x,y)=0$　　记 D 为 $\begin{cases} x=a\cos^3 t \\ y=a\sin^3 t \end{cases}$ 围成面积

$$\therefore S=\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_{\partial D} Pdx+Qdy$$

$$=\int_{\partial D} xdy=\int_0^{2\pi} a\cos^3 t\cdot 3a\sin^2 t\cos t\,dt=3a^2\int_0^{2\pi}\cos^4 t\sin^2 t\,dt=\frac{3}{8}a^2\pi$$

2. 记 Ω: $\begin{cases} 0\le z\le\sqrt{x^2+y^2} \\ x^2+y^2\le h^2 \end{cases}$　　D_1: $\begin{cases} x^2+y^2\le h^2 \\ z=0 \end{cases}$ (上侧)　　D_2: $\begin{cases} x^2+y^2=h^2 \\ 0\le z\le h \end{cases}$ (内侧)



记 $I=\iint_S (y-z)dydz+(z-x)dzdx+(x-y)dxdy$.

$$\therefore I+\iint_{D_1}(y-z)dydz+(z-x)dzdx+(x-y)dxdy+\iint_{D_2}(y-z)dydz+(z-x)dzdx+(x-y)dxdy$$

$$=\iiint_\Omega 0dydz+0dzdx+0dxdy=0$$

另：由对称性

$$\iint_{D_1}(y-z)dydz+(z-x)dzdx+(x-y)dxdy=\iint_{D_1}(x-y)dxdy=\iint_{-D_2}(x-y)dxdy$$

$$\iint_{D_2}(y-z)dydz+(z-x)dzdx+(x-y)dxdy=\iint_{D_2}(x-y)dxdy$$

$\therefore I+0=0$　　　$\therefore I=0$

3. (1) 原式 $=\iint_S (2x^2y-2x^2y)dydz+(2xy^2-2xy^2)dzdx+(4xyz-4xyz)dxdy=0$

(2) 原式 $=\iint_S (-1-1)dydz+(-1-1)dzdx+(-1-1)dxdy=-6\iint_{\sigma_{xy}}dxdy$

S的单位外法向量 $\vec{n}=\left(\frac{\sqrt3}{3},\frac{\sqrt3}{3},\frac{\sqrt3}{3}\right)$　$\therefore \iint_{\sigma_{xy}}dxdy=\frac{\sqrt3}{3}\pi a^2$　　$\therefore$ 原式 $=-2\sqrt3\pi a^2$

练习13.7

1.

(1) 令 $P=2y^2-3x$, $Q=-4xy$.

$\frac{\partial P}{\partial y}=4y$, $\frac{\partial Q}{\partial x}=-4y$. $\frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$ 全微分 ×

(2) 令 $P=2xy+x^2$ $Q=x^2$

$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=2x$. 全微分 √

$\because d(x^2y)=2xy\,dx+x^2dy$, $d(\frac{x^3}{3})=x^2dx$

$\therefore u(x,y)=x^2y+\frac{x^3}{3}+C$

(3) 令 $P=xe^{xy}\sin y$ $Q=e^{xy}\cos y+y$

$\frac{\partial P}{\partial y}=x(xe^{xy}\sin y+e^{xy}\cos y)$ $\frac{\partial Q}{\partial x}=ye^{xy}\cos y$ 全微分 ×

2.

证:

(不确定). 令 $u=\ln\sqrt{x^2+y^2}$. 易证 u 在 $R^2/\{0\}$ 上连续可微

且 $du=\frac{x}{x^2+y^2}dx+\frac{y}{x^2+y^2}dy$

$\therefore$ u 在 $\mathbb{R}^2\backslash\{0\}$ 内积分与路径无关

习题13(A)

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1. (1) 令 $\begin{cases} x=a\cos^3\theta \\ y=a\sin^3\theta \end{cases}$ $x'(\theta)=-3a\cos^2\theta\sin\theta$ $y'(\theta)=3a\sin^2\theta\cos\theta$ $ds=3a|\sin\theta\cos\theta|d\theta$

$\therefore \int_L (x^{\frac{4}{3}}+y^{\frac{4}{3}})ds=\int_0^{2\pi}(a^{\frac{4}{3}}\cos^4\theta+a^{\frac{4}{3}}\sin^4\theta)\cdot 3a|\sin\theta\cos\theta|d\theta$

$=3a^{\frac{7}{3}}\int_0^{2\pi}(\sin^4\theta+\cos^4\theta)|\sin\theta\cos\theta|d\theta$

令 $f(\theta)=(\sin^4\theta+\cos^4\theta)\sin\theta\cos\theta$.

$\therefore \int f d\theta=\int[\sin^4\theta+(1-\sin^2\theta)^2]\sin\theta d\sin\theta \triangleq \int(2t^5-2t^3+t)dt$

$=\frac{t^6}{3}-\frac{t^4}{2}+\frac{t^2}{2}+C=\frac{\sin^6\theta}{3}-\frac{\sin^4\theta}{2}+\frac{\sin^2\theta}{2}+c$

$\therefore$ 原式 $=3a^{\frac{7}{3}}\left(\int_0^{\frac{\pi}{2}}fd\theta+\int_{\pi}^{\frac{3}{2}\pi}fd\theta-\int_{\frac{\pi}{2}}^{\pi}fd\theta-\int_{\frac{3}{2}\pi}^{2\pi}fd\theta\right)=4a^{\frac{7}{3}}$

(2) $x'(t)=-a\sin t$ $y'(t)=a\cos t$ $z'(t)=b$

$ds=\sqrt{a^2\sin^2t+b^2\cos^2t+b^2}dt=\sqrt{a^2+b^2}dt$

$\therefore \int_L(x^2+y^2+z^2)ds=\int_0^{2\pi}(a^2+b^2t^2)\sqrt{a^2+b^2}dt$

$=\sqrt{a^2+b^2}\int_0^{2\pi}(a^2+b^2t^2)dt=2\pi\sqrt{a^2+b^2}(a^2+\frac{4}{3}\pi^2b^2)$

(3) $x'(t)=\cos t-t\sin t$ $y'(t)=\sin t+t\cos t$ $z'(t)=1$

$ds=\sqrt{\cos^2t+t^2\sin^2t+\sin^2t+t^2\cos^2t+1}=\sqrt{2+t^2}dt$

$\therefore \int_L zds=\int_0^{t_0}t\sqrt{2+t^2}dt=\frac{1}{2}\int_0^{t_0^2}\sqrt{2+t^2}dt^2=\frac{(t_0^2+2)^{\frac{3}{2}}-2\sqrt{2}}{3}$

(4) 由轮换对称性. $\int_L x^2ds=\int_L y^2ds=\int_L z^2ds$

$=\frac{1}{3}\int_L(x^2+y^2+z^2)ds=\frac{a}{3}\int_L ds=\frac{2}{3}\pi a^3$

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2.

质心求法：对于某个物块 Ω. 其密度函数为 $\rho(x,y,z)$. 设质心为 $(\bar{x},\bar{y},\bar{z})$

则
$$\begin{cases} \bar{x} = \dfrac{\iiint_\Omega \rho\cdot x\,dv}{\iiint_\Omega \rho\,dv} \\ \bar{y} = \dfrac{\iiint_\Omega \rho\cdot y\,dv}{\iiint_\Omega \rho\,dv} \\ \bar{z} = \dfrac{\iiint_\Omega \rho\cdot z\,dv}{\iiint_\Omega \rho\,dv} \end{cases}$$

设曲线 L: $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

$$x'(t) = \frac{a}{2},\ y'(t) = at,\ z'(t) = at^2$$

$$ds = \sqrt{\frac{a^2}{4} + a^2t^2 + a^2t^4} = \frac{a}{2}(2t^2+1)\,dt$$

$$\int_L x\,ds = \int_0^1 \frac{at}{2}\cdot\frac{a}{2}(2t^2+1)\,dt = \frac{a^2}{4}\int_0^1 (2t^3+t)\,dt = \frac{a^2}{4}$$

$$\int_L y\,ds = \int_0^1 \frac{a}{2}t^2\cdot\frac{a}{2}(2t^2+1)\,dt = \frac{a^2}{4}\int (2t^4+t^2)\,dt = \frac{11}{60}a^2$$

$$\int_L z\,ds = \int_0^1 \frac{a}{3}t^3\cdot\frac{a}{2}(2t^2+1)\,dt = \frac{a^2}{6}\int (2t^5+t^3)\,dt = \frac{7}{72}a^2.$$

L 弧长 $= \int_L ds = \frac{a}{2}\int_0^1 (2t^2+1)\,dt = \frac{5}{6}a$.

$$\therefore\ \bar{x} = \frac{\rho\cdot\int_L x\,ds}{\rho\cdot\int_L ds} = \frac{3}{10}a \qquad \bar{y} = \frac{\rho\cdot\int_L y\,ds}{\rho\cdot\int_L ds} = \frac{11}{50}a \qquad \bar{z} = \frac{\rho\cdot\int_L z\,ds}{\rho\cdot\int_L ds} = \frac{7}{60}a$$

质心 $\left(\frac{3}{10}a,\ \frac{11}{50}a,\ \frac{7}{60}a\right)$

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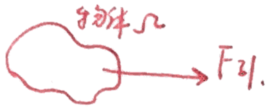
3. 引力求法:



$\rho=\rho(x,y,z)$

物体 m_0

(a,b,c).

记 $r=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

则 $F_x = Gm_0\iiint_\Omega \frac{x-a}{r^3}\rho\, dv$

$F_y = Gm_0\iiint_\Omega \frac{y-b}{r^3}\rho\, dv$

$F_z = Gm_0\iiint_\Omega \frac{z-c}{r^3}\rho\, dv$

$\vec{F} = \vec{F_x}+\vec{F_y}+\vec{F_z}$

解: 金属线在直角坐标下方程为 $x^2+y^2=R^2\ (y\geq 0)$. 令 $\begin{cases} x=R\cos\theta \\ y=R\sin\theta \end{cases}$ $\theta\in[0,\pi]$

$F_x = Gm_0\rho\cdot\int_L \frac{x}{(\sqrt{x^2+y^2})^3}ds$, $F_y = Gm_0\rho\int_L \frac{y}{(\sqrt{x^2+y^2})^3}ds$.

又: $ds = R\,d\theta$

$\therefore F_x = Gm_0\rho\cdot\int_0^\pi \frac{\cos\theta}{R}d\theta = 0$ $\quad F_y = Gm_0\rho\int_0^\pi \frac{\sin\theta}{R}d\theta = \frac{2Gm_0\rho}{R}$

$\therefore F = F_y = \frac{2k}{R}$ ($k = Gm_0\rho$ 为常数)

4.

(1) 记 L: $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \end{cases}$

$\therefore \int_L x^2 y\,dx = \int_{2\pi}^{0} (a\cos\theta)^2 a\sin\theta \, d(a\cos\theta) = a^4 \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta$

$= a^4 \int_0^{2\pi} \frac{1-\cos 4\theta}{8} d\theta = \frac{a^4}{32} \int_0^{8\pi} (1-\cos 4\theta) d(4\theta) = \frac{\pi}{4} a^4$

(2) 记 L: $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 1 \end{cases}$

$\int_L$ 原式 $= \int_0^{2\pi} -3\sin\theta \, d\cos\theta + 2\cos\theta \, d\sin\theta + 2\, d1$

$= 4\pi + \int_0^{2\pi} (1+\sin^2\theta) d\theta = 4\pi + \frac{1}{2}\int_0^{4\pi} \frac{1-\cos 2\theta}{2} d(2\theta) = 4\pi + \pi = 5\pi$

(3) 记 L: $\begin{cases} x = e^t \\ y = e^{-t} \\ z = t \end{cases}$ $0 \le t \le 1$.

原式 $= \int_0^1 e^{-t} de^t - e^t de^{-t} + (e^{2t} + e^{-2t}) dt$

$= \int_0^1 (e^{2t} + e^{-2t} + 2)\, dt = \frac{e^2 - e^{-2}}{2} + 2$

(4) 根据题意 $\Rightarrow \begin{cases} y = \sqrt{2x - x^2}; \\ z^2 = 4 - x^2 - y^2 = 4 - 2x. \end{cases}$

原式 $= \int_L \sqrt{2x^2 - x^2}\, dx - \frac{x-1}{2} dy^2 + \frac{(2x - x^2)}{2} dz^2$

$= \int_L \sqrt{1-(x-1)^2}\, dx - \frac{x-1}{2} d(2x - x^2) + \frac{2x - x^2}{2} d(4-2x)$

$= \int_L \sqrt{1-(x-1)^2}\, dx + (x-1)^2 dx + (x^2 - 2x) dx$

$= \int_1^{-1} \sqrt{1-(x-1)^2}\, d(x-1) + \int_2^0 (2x^2 - 4x + 1) dx$

$= \left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\arcsin(t)\right]\Big|_1^{-1} + \left(\frac{2}{3}x^3 - 2x^2 + x\right)\Big|_2^0$

$= \frac{2}{3} - \frac{\pi}{2}$

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5、证明:

将L分成数段 $A_0, A_1, \dots, A_n$. 取 $M_i \in \widehat{A_{i-1}A_i}$ $(i=1,\dots,n)$ M_i坐标为(ξ_i,η_i)

$$\therefore \int_L Pdx+Qdy = \int_L (P(x,y),Q(x,y))\,d\vec{l} = \lim_{\Delta(T)\to 0}\sum_{i=1}^{n}|(P(\xi_i,\eta_i),Q(\xi_i,\eta_i))|\cdot|\overrightarrow{A_{i-1}A_i}|\cdot\cos\theta_i$$

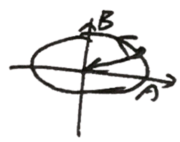
$$\le \lim_{\Delta(T)\to 0}\sum_{i=1}^{n}|(P(\xi_i,\eta_i),Q(\xi_i,\eta_i))||\overrightarrow{A_{i-1}A_i}|$$

$$\le M\cdot\lim_{\Delta(T)\to 0}\sum_{i=1}^{n}|\overrightarrow{A_{i-1}A_i}| = |L|M$$

同理原式 $\ge -|L|M$ $(\cos\theta_i \ge -1)$

$$\therefore \left|\int_L Pdx+Qdy\right| \le |L|M$$

6. 设点(x,y)在$\widehat{AB}$上. 则 $\vec{F}(x,y) = k\cdot\sqrt{x^2+y^2}\,(-x,-y)$



$$\therefore W = \int_{\widehat{AB}} \vec{F}\,d\vec{l}$$

$$= -k\int_{\widehat{AB}} x\sqrt{x^2+y^2}\,dx + y\sqrt{x^2+y^2}\,dy$$

$$= -\frac{k}{2}\int_{\widehat{AB}} \sqrt{x^2+y^2}\,dx^2 + \sqrt{x^2+y^2}\,dy^2$$

$$= -\frac{k}{2}\int_{\widehat{AB}} \sqrt{x^2+y^2}\,d(x^2+y^2)$$

$$= -\frac{k}{2}\int_{b^2}^{a^2}\sqrt{r}\,dr = \frac{k}{3}(a^2-b^2)$$

7. 解：引入球坐标系 $\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$

代入 $\sqrt{x^2+y^2}\cosh(\arctan\frac{y}{x}) = a$， $a\sin\varphi\cosh\theta = a$

$\therefore \sin\varphi = \frac{1}{\cosh\theta}$

$\therefore \begin{cases} x = \frac{a\cos\theta}{\cosh\theta} \\ y = \frac{a\sin\theta}{\sinh\theta} \\ z = a\tanh\theta \end{cases}$ $\theta: 0 \to \theta_0$ $(\theta_0 \in (0, \frac{\pi}{2})$，其中 $z_0 = a\tanh\theta)$

解得 $I = 2a^2(a - \sqrt{a^2 - z_0^2})$

8. (1) 引入柱坐标系 $\begin{cases} x = 2\cos\theta \\ y = y \\ z = 2\sin\theta \end{cases}$ $\therefore \begin{cases} E = x_\theta^2 + y_\theta^2 + z_\theta^2 = 4 \\ F = x_y^2 + y_y^2 + z_y^2 = 1 \\ G = 0 \end{cases}$ $\therefore \sqrt{EF - G^2} = 2$

$\therefore \iint_S xyz\,ds = \iint_D 4\sin\theta\cos\theta \cdot y \cdot 2$

$= 2\int_0^1 y\,dy \int_0^{\frac{\pi}{2}} 2\sin 2\theta\,d\theta = 2$

(2) 记 $\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = R \end{cases} \Rightarrow \begin{cases} E = x_R^2 + y_R^2 + z_R^2 = 2 \\ F = x_\theta^2 + y_\theta^2 + z_\theta^2 = R^2 \\ G = -R\sin\theta\cos\theta + R\sin\theta\cos\theta = 0 \end{cases}$ $\sqrt{EF - G^2} = \sqrt{2}R$

记 $D_1 = [0, 2\pi] \times [0, 1]$ $D_2: \begin{cases} x^2 + y^2 \le 1 \\ z = 1 \end{cases}$

$\therefore \iint_S (x^2+y^2)ds = \iint_{D_1} (x^2+y^2)ds + \iint_{D_2} (x^2+y^2)ds$

$\iint_{D_1} (x^2+y^2)ds = \sqrt{2}\iint_{D_1} R^3 d\theta dR = \sqrt{2}\int_0^{2\pi} d\theta \int_0^1 R^3 dR = \frac{\sqrt{2}}{2}\pi$

$\iint_{D_2} (x^2+y^2)ds = \iint_{D_1} R^3 d\theta dR = \frac{\pi}{2}$

原式 $= \frac{1+\sqrt{2}}{2}\pi$

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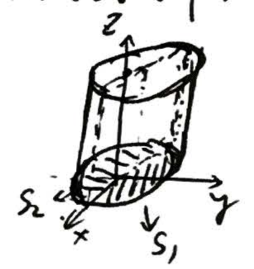
9. 建立空间直角坐标系、使圆柱面的对称轴与z轴重合，原点为其底部中心



S关于xoz、yoz平面对称

$$\therefore F_x = G m_0 \rho \iint_S \frac{x}{r^3} dS = 0$$

$$F_y = G m_0 \rho \iint_S \frac{y}{r^3} dS = 0$$

$$S = S_1 + S_2$$

S_1: $y = \sqrt{R^2 - x^2}$ $(x,z) \in D$, $D = \{(x,z) \mid 0 \le z \le h, -R \le x \le R\}$

$$dS = \sqrt{1 + \left(\frac{x}{\sqrt{R^2-x^2}}\right)^2 + 0}\, dx dz = \frac{R\, dx dz}{\sqrt{R^2 - x^2}}$$

S_2: $y = -\sqrt{R^2 - x^2}$, $(x,z) \in D$

由对称性 $\iint_S \frac{z}{r^3} dS = 2\iint_{S_1} \frac{z}{(\sqrt{x^2+y^2+z^2})^3} dS = 2\int_{-R}^{R} dx \int_0^h \frac{z}{\sqrt{R^2+z^2}} \cdot \frac{R}{\sqrt{R^2-x^2}} dz$

$$= 2\pi R \left(\frac{1}{R} - \frac{1}{\sqrt{R^2+h^2}}\right)$$

$$\therefore F = F_z = G m_0 \rho \iint_S \frac{z}{r^3} dS = 2 G m_0 \rho \pi R \left(\frac{1}{R} - \frac{1}{\sqrt{R^2+h^2}}\right)$$

10. $I_z = \iint_S (x^2+y^2)\rho\, dS$ 其中S: $z = \sqrt{a^2 - x^2 - y^2}$ $(x,y) \in D$, $D = \{(x,y) \mid x^2 + y^2 \le a^2\}$

$$\therefore I_z = \rho \iint_S (x^2+y^2) dS = \rho \iint_D (x^2+y^2) \frac{a}{\sqrt{a^2-x^2-y^2}}$$

$$= \frac{4}{3}\rho \pi a^4$$

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11.

(1) 令 $P(x,y)=xy^2$，$Q(x,y)=2x^2y$，D: $4x^2+9y^2\le 36$.

$$\therefore \int_L xy^2dx+2x^2y\,dy=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=6\iint_D xy\,dxdy.$$

令 $\begin{cases}x=3r\cos\theta\\ y=2r\sin\theta\end{cases}$ 原式 $=6\iint_{D'}6r^3\sin\theta\cos\theta\,d\theta$

$$=18\int_0^1 r^3dr\int_0^{2\pi}\sin 2\theta\,d\theta=0.$$

(2). 令 $Q(x,y)=x^2+2y^2$. D: $(x-2)^2+y^2\le 1$

$$\therefore \int_L(x^2+2y^2)dy=\iint_D\left(\frac{\partial Q}{\partial x}\right)dxdy=2\iint_D dxdy$$

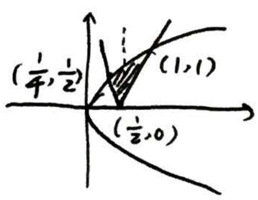
令 $\begin{cases}x=r\cos\theta+2\\ y=r\sin\theta\end{cases}$ 原式 $=2\int_0^1dr\int_0^{2\pi}(r\cos\theta+2)r\,d\theta=8\pi.$

(3) 令 $P(x,y)=x^2y$，$Q(x,y)=-y^2x$.

$$\therefore \int_L Pdx+Qdy=-\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=+\iint_D(x^2+y^2)dxdy.$$

令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ 原式 $=\int_0^a dr\int_0^{\pi}r^3d\theta=\frac{a^4}{4}\pi$

(4) 令 $P(x,y)=x-y^2$，$Q(x,y)=x^2y$.



$$\therefore \int_L Pdx+Qdy=\iint_D(2xy+2y)dxdy$$

$$=\int_{\frac{1}{4}}^{\frac{1}{2}}dx\int_{1-2x}^{\sqrt{x}}y(2x+2)dy+\int_{\frac{1}{2}}^{1}dx\int_{2x-1}^{\sqrt{x}}(2x+2)y\,dy$$

$$=\left(-x^4+\frac{x^3}{3}+2x^2-x\right)\Big|_{\frac{1}{4}}^{1}=\frac{117}{256}$$

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12. (1)

证: $\frac{\partial H(x(t),y(t))}{\partial t} = H_x'(x(t),y(t))\cdot x'(t) + H_y'(x(t),y(t))\cdot y'(t)$

$= x'(t)y'(t) - x'(t)y'(t) = 0$

$\because H \in C^2(\mathbb{R}^2)$ $\therefore H(x(t),y(t)) = \int H' dt = C$

(2) $\because$ 对 $\forall \lambda > 0$, $H(\lambda x, \lambda y) = \lambda^2 H(x,y)$

两边对 λ 求导: $\frac{\partial H(\lambda x,\lambda y)}{\partial(\lambda x)}\cdot\frac{\partial(\lambda x)}{\partial(\lambda)} + \frac{\partial H(\lambda x,\lambda y)}{\partial(\lambda y)}\cdot\frac{\partial(\lambda y)}{\partial(\lambda)} = 2\lambda H(x,y)$

取 $\lambda = 1$ $\therefore x\cdot H_x(x,y) + y\cdot H_y(x,y) = 2H(x,y)$

由格林公式. $S = \iint_{D} ds = \frac{1}{2}\int_{\partial D} x dy - y dx$

$= \frac{1}{2}\int_{\partial D} x(t)y'(t) - y(t)x'(t)\, dt = \frac{1}{2}\int_{\partial D}[x(t)\cdot H_x(x,y) + y(t)\cdot H_y(x,y)]dt$

$= \int_{\partial D} H(x,y) dt$

由(1) $H(x(t),y(t)) = H(x(0),y(0)) = 1$ $(t\in[0,T])$

$\therefore$ 上式 $= \int_{\partial D} 1 dt = \int_0^T 1 dt = T.$

13. 令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $(\theta, r) \in D_1 = [0, 2\pi] \times [0, 1]$.

$\therefore \begin{cases} \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta \end{cases} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cos\theta - \frac{\partial f}{\partial \theta}\cdot\frac{\sin\theta}{r} \\ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \theta}\cdot\frac{\cos\theta}{r} + \frac{\partial f}{\partial r}\sin\theta \end{cases}$

$\therefore \iint_D \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = \iint_{D_1} \frac{\partial f}{\partial r} r^2 d\theta\, dr = \int_0^1 r\,dr \int_0^{2\pi} \frac{\partial f}{\partial r} r\,d\theta.$ $\because ds = r\,d\theta$

$\therefore$ 上式 $= \int_0^1 r\,dr \int_P \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) ds$ (由 green 公式:)

$= \int_0^1 r\,dr \iint_{D_2} e^{-x^2-y^2} dx\,dy$ (其中 $P = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = r^2\}$)
$D_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le r^2\}$

又 $\int_0^{2\pi} \frac{\partial f}{\partial r} r\,d\theta = \int_0^{2\pi} \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) d\theta = \int_P \left(-\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial y}dy\right)$

$= \iint_{D_2} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx\,dy = \iint_{D_2} e^{-x^2-y^2} dx\,dy = \pi(1-e^{-r^2})$

$\therefore \iint_D \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) dx\,dy = \int_0^1 \pi r(1-e^{-r^2}) dr = \frac{\pi}{2e}$

14.(1) 记 $P(x,y)=\dfrac{-y}{ax^2+2bxy+cy^2}$, $Q(x,y)=\dfrac{x}{ax^2+2bxy+cy^2}$ 奇点为$(0,0)$

(必须"打洞")

当$(x,y)\neq(0,0)$时, $P,Q\in C^1$

且 $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}=\dfrac{-ax^2+cy^2}{(ax^2+2bxy+cy^2)^2}$

设Γ: $ax^2+2bxy+cy^2=\varepsilon^2$ 的正向. 其中ε充分小 s.t. Γ包含于L内部.

(设法简化后续计算)

设L与Γ所围成的区域为D'. 则$L+\Gamma^-$为D'的正向边界. (负向)

设Γ围成的区域为D. 则Γ为D的正向边界.

由格林公式(满足连续性条件)

$$\int_L+\int_{\Gamma^-}\frac{-ydx+xdy}{ax^2+2bxy+cy^2}=\iint_{D'}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0.$$

$$\therefore \int_L 原式=\int_\Gamma\frac{-ydx+xdy}{ax^2+2bxy+cy^2}=\int_\Gamma\frac{-ydx+xdy}{\varepsilon^2}=\frac{1}{\varepsilon^2}\int_\Gamma -ydx+xdy.$$

由格林公式. $\dfrac{1}{\varepsilon^2}\displaystyle\int_\Gamma -ydx+xdy=\dfrac{2}{\varepsilon^2}\iint_D dxdy=\dfrac{2S(D)}{\varepsilon^2}$

以下计算$S(D)$:

注意到 $ax^2+2bxy+cy^2=(x,y)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ 作单位正交变换 $\begin{pmatrix}x\\y\end{pmatrix}=T\begin{pmatrix}u\\v\end{pmatrix}$

使得 $T^{-1}\begin{pmatrix}a&b\\b&c\end{pmatrix}T=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$, 其中$\lambda_1,\lambda_2$为$\begin{pmatrix}a&b\\b&c\end{pmatrix}$的特征值.

即为 $\lambda^2-(a+c)\lambda+ac-b^2=0$的两根. $\therefore \lambda_1\lambda_2=ac-b^2$

在T的作用下: $\Gamma\Rightarrow\lambda_1u^2+\lambda_2v^2=\varepsilon^2$.

化为椭圆C: $\dfrac{u^2}{\left(\frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2}+\dfrac{v^2}{\left(\frac{\varepsilon}{\sqrt{\lambda_2}}\right)^2}=1$.

$$\therefore S(D)=S(C)=\pi\cdot\frac{\varepsilon}{\sqrt{\lambda_1}}\cdot\frac{\varepsilon}{\sqrt{\lambda_2}}=\frac{\pi\varepsilon^2}{\sqrt{ac-b^2}}$$

$$\therefore \frac{1}{\varepsilon^2}\int_\Gamma -ydx+xdy=\frac{2}{\varepsilon^2}S(D)=\frac{2\pi}{\sqrt{ac-b^2}}=I$$

14. (2). $\int_L \frac{-Ydx+XdY}{X^2+Y^2} = \int_L \frac{-(cx+dy)(adx+bdy)+(ax+by)(cdx+d\cdot dy)}{(ax+by)^2+(cx+dy)^2}$

$= (ad-bc)\int_L \frac{-ydx+xdy}{(ax+by)^2+(cx+dy)^2}$ 记 $P(x,y)=\frac{-y}{X^2+Y^2}$, $Q(x,y)=\frac{x}{X^2+Y^2}$

$\therefore \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{(b^2+d^2)y^2-(a^2+c^2)x^2}{[(ax+by)^2+(cx+dy)^2]}$

设 Γ: $(ax+by)^2+(cx+dy)^2=\varepsilon^2$ 曲线的正向.（ε充分小使Γ在L内部）

设L与Γ围成区域为D'，$L+\Gamma^-$为D'正向边界，Γ围成区域为D.

根据格林公式

$\int_L+\int_{\Gamma^-} \frac{-Ydx+XdY}{X^2+Y^2} = (ad-bc)\int_L+\int_{\Gamma^-} \frac{-ydx+xdy}{(ax+by)^2+(cx+dy)^2} = \iint_{D'}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=0.$

$\therefore \int_L \frac{-Ydx+XdY}{X^2+Y^2} = (ad-bc)\int_\Gamma \frac{-ydx+xdy}{(ax+by)^2+(cx+dy)^2} = \frac{ad-bc}{\varepsilon^2}\int_\Gamma -ydx+xdy.$

由格林公式 $\int_\Gamma -ydx+xdy = 2\iint_D dxdy = 2S(D).$

以下计算 $S(D)$:

注意到 $(ax+by)^2+(cx+dy)^2=(a^2+c^2)x^2+(b^2+d^2)y^2+2(ab+cd)xy=(x\ y)\begin{pmatrix}a^2+c^2 & ab+cd\\ ab+cd & a^2+c^2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

作单位正交变换 $T\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ 使 $T^{-1}\begin{pmatrix}a^2+c^2 & ab+cd\\ ab+cd & b^2+d^2\end{pmatrix}T=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$. 其中$\lambda_1$、$\lambda_2$为$\begin{pmatrix}a^2+c^2 & ab+cd\\ ab+cd & b^2+d^2\end{pmatrix}$两特征值

$\therefore \lambda_1$、λ_2为方程 $\lambda^2-(a^2+b^2+c^2+d^2)\lambda+(ad-bc)^2$ 的两根　故 $\lambda_1\lambda_2=(ad-bc)^2$

在T作用下，$\Gamma \Rightarrow \lambda_1u^2+\lambda_2v^2=\varepsilon^2$. 化为椭圆C: $\frac{u^2}{(\frac{\varepsilon}{\sqrt{\lambda_1}})^2}+\frac{v^2}{(\frac{\varepsilon}{\sqrt{\lambda_2}})^2}=1$

$\therefore S(D)=S(C)=\pi\cdot\frac{\varepsilon}{\sqrt{\lambda_1}}\cdot\frac{\varepsilon}{\sqrt{\lambda_2}}=\frac{\pi\varepsilon^2}{\sqrt{\lambda_1\lambda_2}}=\frac{\pi\varepsilon^2}{|ad-bc|}$

$\therefore \int_L \frac{-Ydx+XdY}{X^2+Y^2} = \frac{ad-bc}{\varepsilon^2}\cdot\frac{2\pi\varepsilon^2}{|ad-bc|} = 2\pi\cdot\frac{ad-bc}{|ad-bc|} = 2\pi\cdot sgn(ad-bc).$

15.

(1) 记 $V: x^2+y^2+z^2 \le 1$

$\therefore \iint_S x^3 dydz + y^3 dzdx + z^3 dxdy = 3\iiint_V (x^2+y^2+z^2)\,dxdydz$

令 $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$ $\varphi \in [0,\pi)$ $\theta \in [0,2\pi]$

原式 $= 3\iiint_{V'} r^2 \cdot r^2 \sin\varphi \, dr d\varphi d\theta$

$= 3\int_0^1 r^4 dr \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi$

$= \frac{12}{5}\pi$

(2) 记V为S内部

$\therefore \iint_S (xy\cos\alpha + y^2\cos\beta + y^2\cos\gamma)\,ds$

$= \iiint_V (y+2y)\,dxdydz = 3\int_0^1 dx \int_0^1 y\,dy \int_0^1 dz$

$= \frac{3}{2}$

(3) 记V为S内部

$\therefore \iint_S (x^2 dydz + y^2 dzdx + z^2 dxdy) = 2\iiint_V (x+y+z)\,dxdydz$

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ $\therefore \iiint_V (x+y+z)\,dxdydz = \int_0^3 dr \int_0^{2\pi} d\theta \int_{4-r^2}^{4} r^2(\cos\theta+\sin\theta) + rz \; dz$

$= \int_0^3 dr \int_0^{2\pi} r^4(\cos\theta+\sin\theta) + 4r^3 - \frac{r^5}{2}\,d\theta$

$= \int_0^3 2\pi\left(4r^3 - \frac{r^5}{2}\right)dr = \frac{81}{2}\pi$

$\therefore$ 原式 $= 81\pi$

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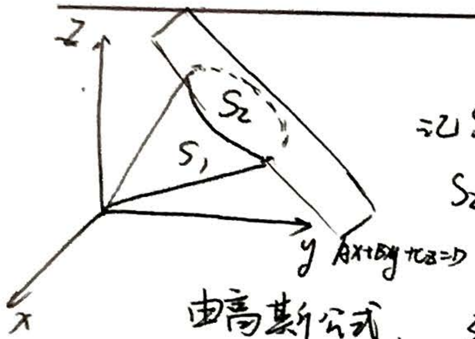
16.



记 S_1 为锥体侧面，单位外法向量 $\vec{n_1}=\frac{1}{\sqrt{F_x^2+F_y^2+F_z^2}}(F_x,F_y,F_z)$

S_2 为锥体底面，单位外法向量 $\vec{n_2}=\frac{1}{\sqrt{A^2+B^2+C^2}}(A,B,C)$

由高斯公式 $3V=\iiint\limits_V 3\,dx\,dy\,dz=\iint\limits_{S_1+S_2} x\,dy\,dz+y\,dz\,dx+z\,dx\,dy$

$$=\iint\limits_{S_1}(x,y,z)\cdot\vec{n_1}\,ds+\iint\limits_{S_2}(x,y,z)\vec{n_2}\,ds$$

由齐次函数的高斯定理（(12) 已证） $xF_x+yF_y+zF_z=3F=0$

$\therefore \iint\limits_{S_1}(x,y,z)\vec{n_1}\,ds=\frac{1}{\sqrt{F_x^2+F_y^2+F_z^2}}\iint\limits_{S_1}(xF_x+yF_y+zF_z)\,ds=0$

$\therefore 3V=\iint\limits_{S_2}(x,y,z)\cdot\vec{n_2}\,ds=\frac{1}{\sqrt{A^2+B^2+C^2}}\iint\limits_{S_2}(Ax+By+Cz)\,ds$

$$=\frac{D}{\sqrt{A^2+B^2+C^2}}\iint\limits_{S_2}ds$$

$$=\frac{D}{\sqrt{A^2+B^2+C^2}}\cdot S$$

由点到平面距离公式

$(0,0,0)$ 到 $Ax+By+Cz=D$ 的距离

$$H=\frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}=\frac{D}{\sqrt{A^2+B^2+C^2}}$$

$\therefore V=\frac{1}{3}SH.$

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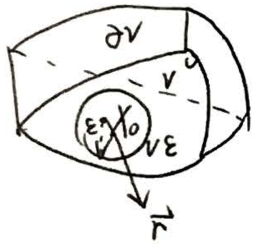
17. 记 $\vec{r}=(x_0,y_0,z_0)$ $\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$

$$\therefore \iint_S \cos(\vec{r},\vec{n})\,dS=\iint_S \frac{x_0\cos\alpha+y_0\cos\beta+z_0\cos\gamma}{|\vec{r}|}\,dS$$

$$=\iint_S \frac{x_0\,dy\,dz+y_0\,dz\,dx+z_0\,dx\,dy}{|\vec{r}|}=\iiint_{S^\circ}\frac{0}{|\vec{r}|}\,dx\,dy\,dz=0$$

18. **错误解法：**



记 $X_0=(\xi,\eta,\zeta)$ 首先，计算

$$\frac{\partial\left(\frac{x-\xi}{r}\right)}{\partial x}=\frac{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}-(x-\xi)\cdot\frac{2(x-\xi)}{2\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}}{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$

$$=\frac{(y-\eta)^2+(z-\zeta)^2}{r^{\frac{3}{2}}}$$

同理 $\frac{\partial\left(\frac{y-\eta}{r}\right)}{\partial y}=\frac{(x-\xi)^2+(z-\zeta)^2}{r^{\frac{3}{2}}}$，$\frac{\partial\left(\frac{z-\zeta}{r}\right)}{\partial z}=\frac{(x-\xi)^2+(y-\eta)^2}{r^{\frac{3}{2}}}$.

$$右式=\frac{1}{2}\iint_{\partial V}\cos(\vec{r},\vec{n})\,dS$$

$$=\frac{1}{2}\iint_{\partial V}\frac{(x-\xi,y-\eta,z-\zeta)\cdot(\cos\alpha,\cos\beta,\cos\gamma)}{r}\,dS$$

$$=\frac{1}{2}\iint_{\partial V}\frac{x-\xi}{r}\,dy\,dz+\frac{y-\eta}{r}\,dz\,dx+\frac{z-\zeta}{r}\,dx\,dy=\frac{1}{2}\iiint_V\frac{2\,dx\,dy\,dz}{r}$$

$$=\iiint_{V_\varepsilon}\frac{dx\,dy\,dz}{r}+\iiint_{V_\varepsilon'}\frac{dx\,dy\,dz}{r}$$

其中 V_ε': $\{(x,y,z)\mid(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2<\varepsilon^2\}$

令 $\varepsilon\to0^+$. 显然 $\iiint_{V_\varepsilon'}\frac{dx\,dy\,dz}{r}\to0$

$$\therefore \frac{1}{2}\iint_{\partial V}\cos(\vec{r},\vec{n})\,dS=\lim_{\varepsilon\to0^+}\iiint_{V_\varepsilon}\frac{dx\,dy\,dz}{r}$$

思考：错在何处？

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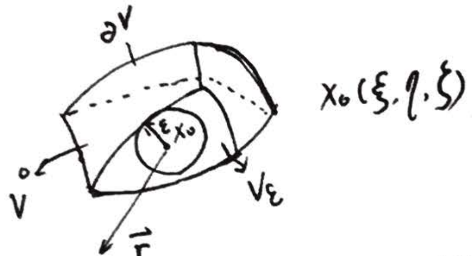
18. 正确解答



首先，经计算: $\frac{\partial\left(\frac{x-\xi}{r}\right)}{\partial x} = \frac{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2} - (x-\xi)\frac{2(x-\xi)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}}{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2} = \frac{(y-\eta)^2+(z-\zeta)^2}{r^{\frac{3}{2}}}$

同理 $\frac{\partial\left(\frac{y-\eta}{r}\right)}{\partial y} = \frac{(x-\xi)^2+(z-\zeta)^2}{r^{\frac{3}{2}}}$, $\frac{\partial\left(\frac{z-\zeta}{r}\right)}{\partial z} = \frac{(x-\xi)^2+(y-\eta)^2}{r^{\frac{3}{2}}}$

$\therefore \frac{1}{2}\iint_{\partial V}\cos(\vec{r},\vec{n})ds = \frac{1}{2}\iint_{\partial V}\frac{(x-\xi,y-\eta,z-\zeta)\cdot(\cos\alpha,\cos\beta,\cos\gamma)}{r}ds$

$= \frac{1}{2}\iint_{\partial V}\frac{x-\xi}{r}dydz + \frac{y-\eta}{r}dzdx + \frac{z-\zeta}{r}dxdy.$

注意此处不能直接应用高斯公式. 因为 (ξ,η,ζ) 显然是瑕点，不满足连续条件.

"挖洞"讨论：当 $(x,y,z)\neq(\xi,\eta,\zeta)$ 时. 可应用高斯公式. $\therefore$ 上式 $= \frac{1}{2}\iint_{\partial V}\frac{\partial\left(\frac{x-\xi}{r}\right)}{\partial x} + \frac{\partial\left(\frac{y-\eta}{r}\right)}{\partial y} + \frac{\partial\left(\frac{z-\zeta}{r}\right)}{\partial z}dxdydz$

$= \iint_{\partial V}\frac{dxdydz}{r}.$

我们记 $B_\varepsilon = V - V_\varepsilon$，$S(\varepsilon)$ 为 $B(\varepsilon)$ 外向表面.

取 ε 充分小使 $B_\varepsilon \subset V$. 在 V_ε 区域上（V 中挖去 B_ε 的区域）应用高斯公式：

$\left(\iint_{\partial V} - \iint_{S(\varepsilon)}\right)\frac{x-\xi}{r}dydz + \frac{y-\eta}{r}dzdx + \frac{z-\zeta}{r}dxdy = \iiint_{V_\varepsilon}\frac{2dxdydz}{r} = \iiint_{V}\frac{2dxdydz}{r} - \iiint_{B_\varepsilon}\frac{2dxdydz}{r}$

在球面 $S(\varepsilon)$ 上，$\vec{r} = \varepsilon\cdot\vec{n}$，$\cos(\vec{r},\vec{n})=1$

而 $\iint_{S(\varepsilon)}\frac{(x-\xi)dydz}{r} + \frac{(y-\eta)dzdx}{r} + \frac{(z-\zeta)dxdy}{r} = \iint_{S(\varepsilon)}\cos(\vec{r},\vec{n})ds = \iint_{S(\varepsilon)}ds = 4\pi\varepsilon^2.$

（此外 $\iiint_{B_\varepsilon}\frac{dxdydz}{r} = \int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{\varepsilon}r\sin\varphi\, dr = 2\pi\varepsilon^2$（作广义球坐标变换））

$\therefore$ 当 $\varepsilon\to 0$ 时 $\lim_{\varepsilon\to 0}\iiint_{V_\varepsilon}\frac{dxdydz}{r} = \frac{1}{2}\iint_{\partial V}\cos(\vec{r},\vec{n})ds.$

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19. (1) 记 l 为线段 AB ($B\to A$ 方向)

$\because L\cup l$ 包围了闭合曲面 S. 由斯托克斯公式,

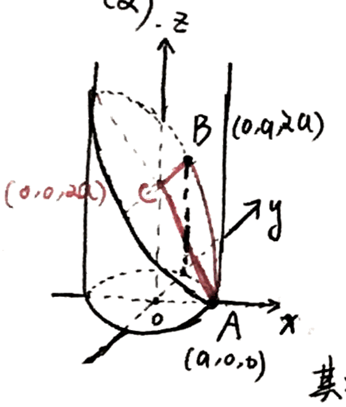
$$I+\int_l (x^2-yz)dx+(y^2-xz)dy+(z^2-xy)dz$$

$$=\iint_S (-x+x)dydz+(-y+y)dzdx+(-z+z)dxdy=0$$

$$\therefore I=-\int_l (x^2-yz)dx+(y^2-xz)dy+(z^2-xy)dz$$

$$=-\int_h^0 z^2dz=\frac{h^3}{3}.$$

(2)



记 L_1 为弧线段 AB ($A\to B$) L_2 为线段 BC $\begin{cases}x=0\\ z=2a\end{cases}$ $y: a\to 0$ L_3 为线段 CA: $\begin{cases}y=0\\ z=2a-2x\end{cases}$ $x: 0\to A$

记 S 为 L_1、L_2、L_3 围成区域

由斯托克斯公式

$$-2\iint_S zdydz+xdzdx+ydxdy=\int_{L_1+L_2+L_3} y^2dx+z^2dy+x^2dz$$

其中: $\int_{L_2} y^2dx+z^2dy+x^2dz=\int_a^0 4a^2dy=-4a^3$ $\int_{L_3} y^2dx+z^2dy+x^2dz=-2\int_0^a x^2dx=-\frac{2}{3}a^3$

而令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\\ z=2a-2r\cos\theta\end{cases}$ 记 $D=[0,a]\times[0,\frac{\pi}{2}]$. 其中 $(r,\theta)\in D$.

$$\therefore \frac{D(y,z)}{D(r,\theta)}=2r \qquad \frac{D(z,x)}{D(r,\theta)}=0 \qquad \frac{D(x,y)}{D(r,\theta)}=r$$

$$\therefore \iint_S zdydz+xdzdx+ydxdy=\iint_D (2a-2r\cos\theta)\cdot 2r+r\sin\theta\cdot r \ drd\theta$$

$$=\iint_D (4ar+r^2\sin\theta-4r^2\cos\theta)drd\theta=\int_0^{\frac{\pi}{2}}d\theta\int_0^a 4ar+r^2(\sin\theta-4\cos\theta)dr$$

$$=(\pi-1)a^3$$

$$\therefore (2-2\pi)a^3=I-4a^3-\frac{2}{3}a^3 \qquad \therefore I=\left(\frac{20}{3}-2\pi\right)a^3$$

20.

(1) $\because \frac{\partial(2xy)}{\partial y} = \frac{\partial(x^2-y^2)}{\partial x} = 2x$, $P=2xy$, $Q=x^2-y^2$ 在 $\mathbb{R}^2$ 上处处成立, $\mathbb{R}^2$ 单连通

$\therefore$ 积分与路径无关.

由于 $x^2y-\frac{y^3}{3}$ 是 $2xydx+(x^2-y^2)dy$ 的一个原函数

$\therefore \int_{(1,1)}^{(x,y)} 2xydx+(x^2-y^2)dy = x^2y-\frac{y^3}{3}\Big|_{(1,1)}^{(x,y)} = x^2y-\frac{y^3+2}{3}$.

(2) $\because \frac{\partial(\sin y)}{\partial y} = \frac{\partial(x\cos y)}{\partial x} = \cos y$ $\mathbb{R}^2$ 是单连通曲域

$\therefore$ 积分与路径无关

由于 $x\sin y$ 是 $\sin y dx + x\cos y dy$ 的一个原函数

$\therefore \int_{(0,0)}^{(x,y)} \sin y dx + x\cos y dy = x\sin y\Big|_{(0,0)}^{(x,y)} = x\sin y$.

21. (1) 记 $P=yz$, $Q=zx$, $R=xy$.

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = z$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = y$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = x$ $\therefore \omega$ 是全微分

显然 $u=xyz$ 是其原函数. $u=xyz+C$.

(2) 记 $P=\sin(yz)$ $Q=zx\cos(yz)$ $R=xy\cos(yz)$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = z\cos(yz)$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = y\cos(yz)$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = x\cos(yz)-xyz\sin(yz)$

$\therefore \omega$ 是全微分

$u=\int Pdx+\int Qdy+\int Rdz = x\sin(yz)+x\sin(yz)-x\sin(yz)+C = x\sin(yz)+C$.

(3) 记 $P(x,y,z)=y$, $Q=z$, $R=x$

$\because \frac{\partial R}{\partial y}=0$, $\frac{\partial Q}{\partial z}=1$ $\frac{\partial R}{\partial y}\neq\frac{\partial Q}{\partial z}$ $\therefore \omega$ 不是全微分

22. 解: 设 $L: x^2+y^2=1$. 取正向.

若 $w=\lambda w_0+du$，则 $\int_L w=\int_L \lambda w_0+du$

由 14(1)(2)，$\int_L w=\frac{2\pi}{\sqrt{ac-b^2}}$，$\int_L \lambda w_0=2\pi\lambda$，$\int_L du=0$ (积分与路径无关, L是闭曲线) 不妨设

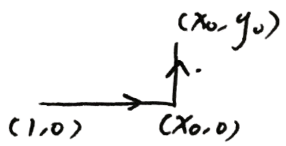
$\therefore \frac{2\pi}{\sqrt{ac-b^2}}=2\pi\lambda$

$\therefore \lambda=\frac{1}{\sqrt{ac-b^2}}$

由 $du=w-\lambda w_0$

对 $\forall(x,y)$ 取一条从 $(1,0)$ 出发. 不过原点的路径 至 (x,y)

$u(x,y)=\int_{(1,0)}^{(x,y)}(w-\lambda w_0)$



其中 $\int_{(1,0)}^{(x,y)} w=\int_{(1,0)}^{(x,0)} w+\int_{(x,0)}^{(x,y)} w$

$$=0+\int_0^{y_0}\frac{x_0\,dy}{ax_0^2+2bx_0y+cy^2}=\int_0^{y_0}\frac{x_0\,dy}{c\left(y+\frac{b}{c}x_0\right)^2+x_0^2\left(a-\frac{b^2}{c}\right)}$$

$$=\frac{1}{\sqrt{ac-b^2}}\arctan\frac{cy+bx}{x\sqrt{ac-b^2}}$$

又: 注意到 $d\left(\arctan\frac{y}{x}\right)=w_0$

$$\therefore u(x,y)=\begin{cases}\frac{1}{\sqrt{ac-b^2}}\left(\arctan\frac{cy+bx}{x\sqrt{ac-b^2}}-\arctan\frac{y}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$$

(经检验，满足积分与路径无关)

习题13-B组题

P295 (B) 6.

知识补充

方向导数 $\frac{\partial f}{\partial \vec{l}} = \nabla f \cdot \vec{l}$

单位外法向量 $\vec{n} = (\cos\alpha, \cos\beta)$

格林公式 $\oint_{\partial D} (P\cos\alpha + Q\cos\beta)\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dxdy$

(1) $\int_{\partial D} v \frac{\partial u}{\partial \vec{n}} = \int_{\partial D} v(u_x \cdot \cos\alpha + u_y \cdot \cos\beta)\, ds = \iint_D \left(\frac{\partial (v u_x)}{\partial x} + \frac{\partial (v u_y)}{\partial y} \right) dxdy$

$= \iint_D (u_x v_x + u_y v_y)\, dxdy + \iint_D v\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial y^2}\, dxdy$

$\therefore \iint_D v \Delta u\, dxdy = -\iint_D \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y}\right) dxdy + \int_{\partial D} v \frac{\partial u}{\partial \vec{n}}\, ds$

(2) u、v 互换. $\iint_D u \Delta v\, dxdy = -\iint \nabla u \cdot \nabla v\, dxdy + \int_{\partial D} u \frac{\partial v}{\partial \vec{n}}\, ds$

与(1)中结论相减 即得证.

P296 — (B) 7.

(1)(2) 与 6 完全类似

(3) 由(2) 记 $B_\varepsilon = \{(x,y,z) \mid (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 < \varepsilon^2\}$. $G_\varepsilon = G \backslash B_\varepsilon$

在 G_ε 上. 由 6(2) 结论

$$\underbrace{\iint_{\partial G_\varepsilon} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial (\frac{1}{r})}{\partial \vec{n}}\right) ds = \iiint_{G_\varepsilon} \left(\frac{1}{r}\Delta u + u\Delta(\tfrac{1}{r})\right) dxdydz}\qquad (*)$$

容易证明 $\Delta(\frac{1}{r}) = 0$ （$\frac{1}{r}$是调和函数）

$\because u \in C^2(G)$. G是有界闭域 $\therefore \Delta u$ 有界, 设 $|\Delta u| \le M$

$\therefore \left|\iiint_{B_\varepsilon} \frac{1}{r}\Delta u\, dxdydz\right| \le M \left|\iiint_{B_\varepsilon} \frac{1}{r}\, dxdydz\right| = 2\pi\varepsilon^2 \to 0 \quad (\varepsilon \to 0)$

$\therefore \iiint_G \left(\frac{1}{r}\Delta u + u\Delta(\tfrac{1}{r})\right) dxdydz = \iiint_G \frac{1}{r}\Delta u\, dxdydz = \left(\iiint_{G_\varepsilon} + \iiint_{B_\varepsilon}\right) \frac{1}{r}\Delta u\, dxdydz$

$$\to \iiint_{G_\varepsilon} \frac{1}{r}\Delta u\, dxdydz \qquad \text{也即} \iiint_{G_\varepsilon} \frac{1}{r}\Delta u\, dxdydz \to \iiint_{G} \frac{1}{r}\Delta u\, dxdydz$$

在 ∂B_ε 上： $\vec{n}=\left(\frac{x-\xi}{r}, \frac{y-\eta}{r}, \frac{z-\zeta}{r}\right)$ 为 B_ε 的单位外法向量， $r=\varepsilon$.

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}} = -\frac{1}{r^2}(\cos^2\alpha+\cos^2\beta+\cos^2\gamma) = -\frac{1}{r^2}$$

$$\therefore \iint_{\partial B_\varepsilon} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds = \frac{1}{\varepsilon}\iint_{\partial B_\varepsilon} \frac{\partial u}{\partial \vec{n}} ds + \frac{1}{\varepsilon^2}\iint_{\partial B_\varepsilon} u\, ds$$

由积分中值定理

$$= \frac{1}{\varepsilon}\cdot 4\pi\varepsilon^2 \cdot \left.\frac{\partial u}{\partial \vec{n}}\right|_{M'} + \frac{1}{\varepsilon^2}\cdot 4\pi\varepsilon^2 \cdot u(N')$$

$\varepsilon\to 0$， M'与$N'\to P(\xi,\eta,\zeta)$

$$\to 4\pi\cdot u(\xi,\eta,\zeta) \quad (\varepsilon\to 0)$$

$$\therefore \iint_{\partial G_\varepsilon} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds = \iint_{\partial G} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds - \iint_{\partial B_\varepsilon} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds$$

$$\to \iint_{\partial G} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds - 4\pi\cdot u(\xi,\eta,\zeta)$$

由(*) $$\iiint_G \frac{1}{r}\Delta u\, dxdydz = \iint_{\partial G} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds - 4\pi\cdot u(\xi,\eta,\zeta)$$

$$\therefore u(\xi,\eta,\zeta) = -\frac{1}{4\pi}\iiint_G \frac{\Delta u}{r} dxdydz + \frac{1}{4\pi}\iint_{\partial G} \left(\frac{1}{r}\frac{\partial u}{\partial \vec{n}} - u\frac{\partial\left(\frac{1}{r}\right)}{\partial \vec{n}}\right) ds$$

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(1) 注意到 $x(x^2+y^2)dx + y(x^2+y^2)dy = d\,\frac{(x^2+y^2)^2}{4}$

记 $m(x,y)=m(x^2+y^2)$ 是一元函数与 x^2+y^2 的复合.

则 $m(x,y)\left[x(x^2+y^2)dx + y(x^2+y^2)dy\right]$ 仍是全微分形式

需使 $m\left(-y\sqrt{x^2+y^2+1}\,dx + x\sqrt{x^2+y^2+1}\,dy\right) = P\,dx+Q\,dy$ 为全微分，需有 $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$

$$\therefore 2m'x^2\sqrt{x^2+y^2+1} + m\left(\sqrt{x^2+y^2+1}+\frac{x^2}{\sqrt{x^2+y^2+1}}\right) = -2m'y^2\sqrt{x^2+y^2+1} - m\left(\sqrt{x^2+y^2+1}+\frac{y^2}{\sqrt{x^2+y^2+1}}\right)$$

令 $x^2+y^2=t$. 上式化为 $2m'(t)\,t\sqrt{t+1} + m(t)\left(\sqrt{t+1}+\frac{t}{\sqrt{t+1}}\right)=0$

$$\Rightarrow 2m'(t)\,t(1+t) + m(t)(3t+2)=0$$

$$\frac{m'(t)}{m(t)} = -\frac{3t+2}{t(t+1)} = -\frac{1}{t} - \frac{1}{2(t+1)}$$

$\Rightarrow m(t) = \dfrac{c}{t\sqrt{t+1}}$ 不妨取 $c=1$，$m(x,y) = \dfrac{1}{(x^2+y^2)\sqrt{x^2+y^2+1}}$

可算得 $u = \arctan\dfrac{y}{x} - \sqrt{x^2+y^2+1} + C$

(2)

记 $m(x,y) = m(ay+bx)$ 是一元函数与 $ay+bx$ 的复合

要使 $m\omega = P\,dx + Q\,dy$ 为全微分，需有 $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

即有 $b m' y\left[(ay+bx)^3 + bx^3\right] + my\left[3b(ay+bx)^2 + 3bx^2\right] = a m' x\left[(ay+bx)^3 + ay^3\right] + mx\left[3a(ay+bx)^2 + 3ay^2\right]$

令 $t = ay+bx$

$\dfrac{m'(t)}{m(t)} = -\dfrac{3}{t} \Rightarrow m(t) = \dfrac{c}{t^3}$ 不妨取 $c=1$，$m = \dfrac{c}{(ay+bx)^3}$

算得 $u(x,y) = \dfrac{x^2+y^2}{2} + \dfrac{x^2y^2}{2(ay+bx)^2} + C$