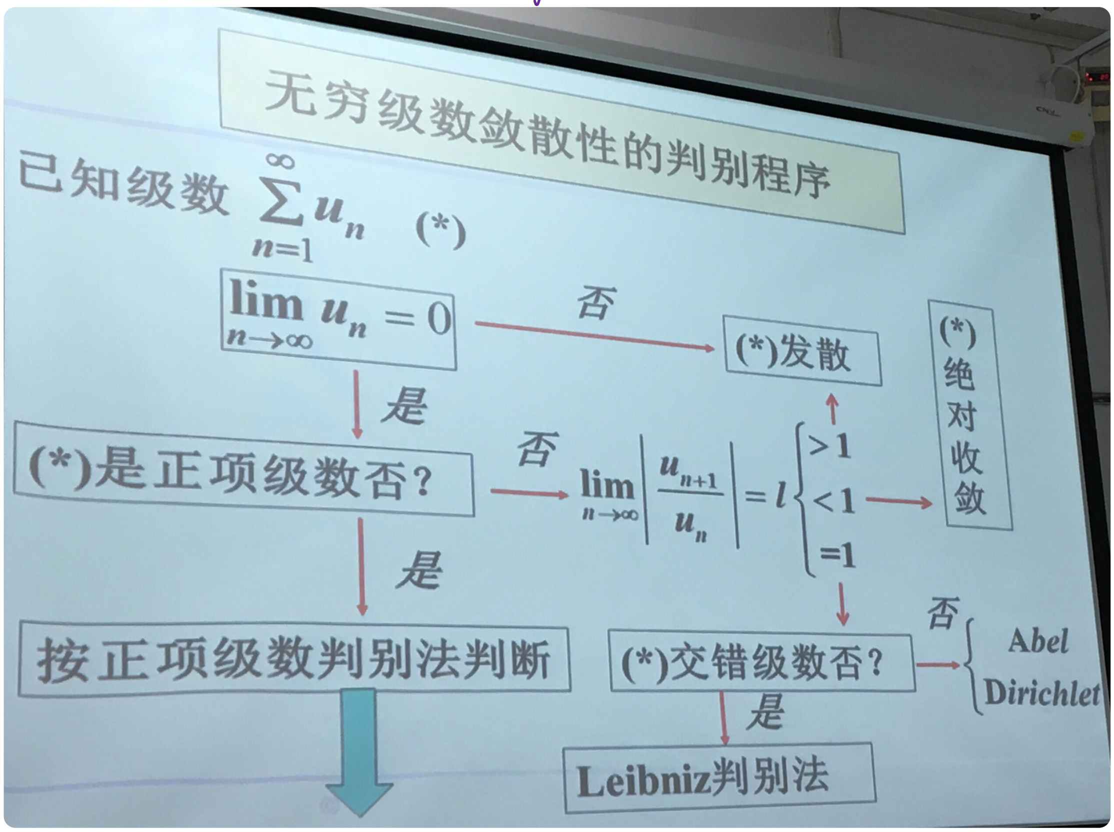
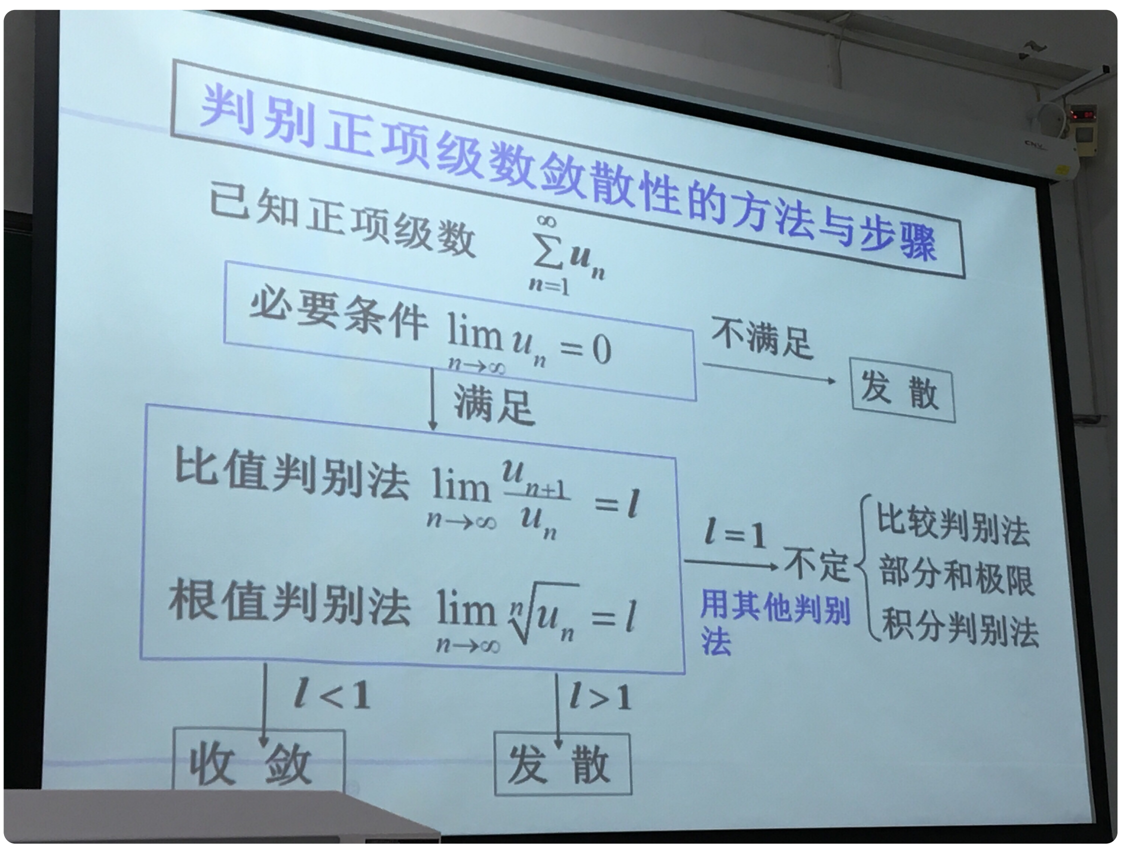
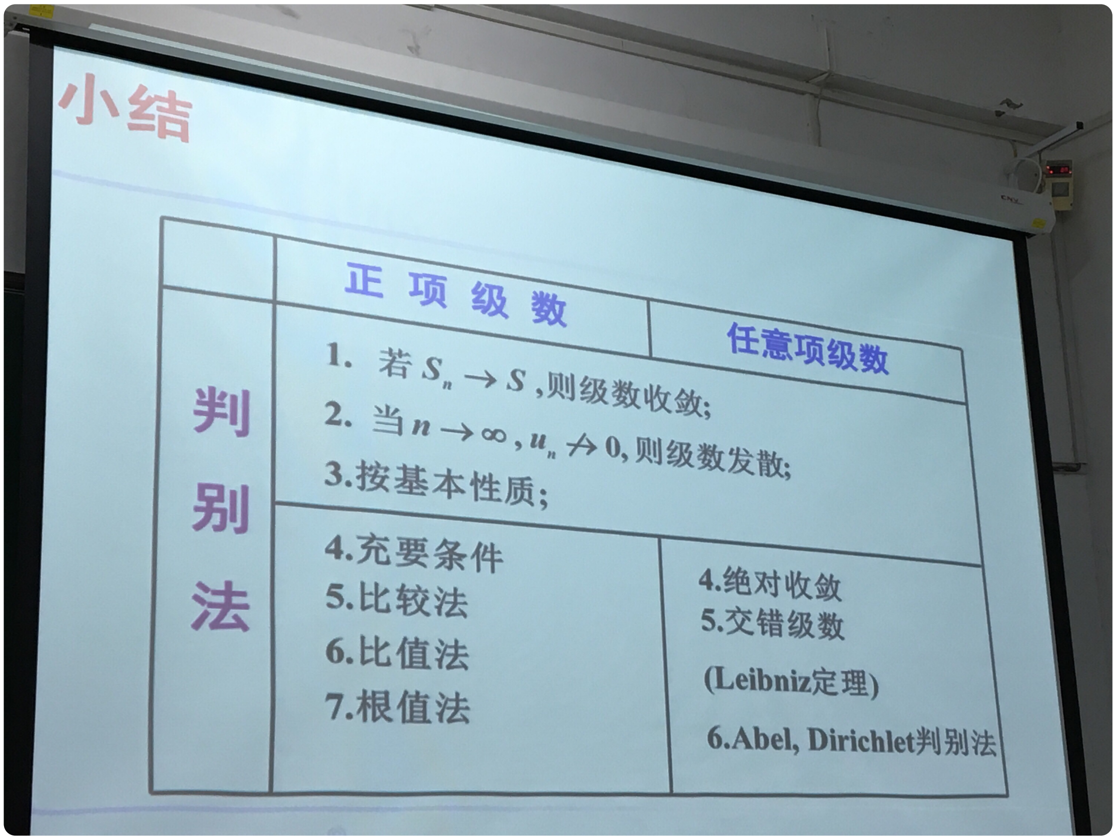
第十四章 数项级数


无穷级数敛散性的判别程序
已知级数 $\sum_{n=1}^{\infty} u_n$ (*)
$\lim_{n\to\infty} u_n = 0$
否
(*)发散
是
(*)是正项级数否？
否
$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l$
>1
<1
=1
(*)绝对收敛
是
按正项级数判别法判断
否
(*)交错级数否？
Abel
Dirichlet
是
Leibniz判别法


判别正项级数敛散性的方法与步骤
已知正项级数 $\sum_{n=1}^{\infty} u_n$
必要条件 $\lim_{n\to\infty} u_n = 0$
不满足
发 散
满足
比值判别法 $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$
根值判别法 $\lim_{n\to\infty} \sqrt[n]{u_n} = l$
$l=1$
不定
用其他判别法
比较判别法
部分和极限
积分判别法
$l<1$
收 敛
$l>1$
发 散


小结
正项级数
任意项级数
判别法
1. 若 $S_n \to S$，则级数收敛;
2. 当 $n \to \infty$, $u_n \not\to 0$, 则级数发散;
3.按基本性质;
4.充要条件
5.比较法
6.比值法
7.根值法
4.绝对收敛
5.交错级数
(Leibniz定理)
6.Abel, Dirichlet判别法

练习14.1

1、(1) $\dfrac{n}{(n+1)(n+2)(n+3)} = \dfrac{1}{(n+2)(n+3)} - \dfrac{1}{(n+1)(n+2)(n+3)} = \left(\dfrac{1}{n+2} - \dfrac{1}{n+3}\right) - \dfrac{1}{2}\left[\dfrac{1}{(n+1)(n+2)} - \dfrac{1}{(n+2)(n+3)}\right]$

$= \dfrac{3}{2}\left(\dfrac{1}{n+2} - \dfrac{1}{n+3}\right) - \dfrac{1}{2}\left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$

$\therefore S_n = \dfrac{3}{2}\left(\dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{4} - \dfrac{1}{5} + \cdots + \dfrac{1}{n+2} - \dfrac{1}{n+3}\right) - \dfrac{1}{2}\left(\dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$

$= \dfrac{3}{2}\left(\dfrac{1}{3} - \dfrac{1}{n+3}\right) - \dfrac{1}{2}\left(\dfrac{1}{2} - \dfrac{1}{n+2}\right) = \dfrac{1}{4} - \dfrac{3}{2(n+3)} + \dfrac{1}{2(n+2)}$

$= \dfrac{1}{4} - \dfrac{2n+3}{2(n+2)(n+3)}$ 　　原式 $= \lim\limits_{n\to\infty} S_n = \dfrac{1}{4}$

(2) $\left.\begin{aligned} S_n &= \dfrac{1}{2} + \dfrac{3}{4} + \dfrac{5}{8} + \cdots + \dfrac{2n-1}{2^n} \\ 2S_n &= 1 + \dfrac{3}{2} + \dfrac{5}{4} + \cdots + \dfrac{2n-1}{2^{n-1}} \end{aligned}\right\} \Rightarrow S_n = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^{n-2}} - \dfrac{2n-1}{2^n}$

$= 1 + \dfrac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} - \dfrac{2n-1}{2^n} = 3 + \dfrac{2n+3}{2^n}$

原式 $= \lim\limits_{n\to\infty} S_n = 3$

(3) $u_n = (\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{n+1} - \sqrt{n})$

$S_n = (\sqrt{3} - \sqrt{2}) - (\sqrt{2} - \sqrt{1}) + \cancel{(\sqrt{4} - \sqrt{3})} - \cancel{(\sqrt{3} - \sqrt{2})} + \cdots + \cancel{(\sqrt{n+1} - \sqrt{n})} - \cancel{(\sqrt{n} - \sqrt{n-1})} + (\sqrt{n+2} - \sqrt{n+1}) - \cancel{(\sqrt{n+1} - \sqrt{n})}$

$= (\sqrt{n+2} - \sqrt{n+1}) - \sqrt{2} + 1$

$= \dfrac{1}{\sqrt{n+2} + \sqrt{n+1}} + 1 - \sqrt{2}$ 　　原式 $= \lim\limits_{n\to\infty} S_n = 1 - \sqrt{2}$

2. $\because \sum\limits_{n=1}^{\infty} a_n$ 与 $\sum\limits_{n=1}^{\infty} b_n$ 收敛

$\therefore$ 对 $\forall \varepsilon > 0$:

$\exists N_1 \in N^*$ s.t $\forall n > N_1, \forall p \in N^*$, $-\varepsilon < a_{n+1} + \cdots + a_{n+p} < \varepsilon$

$\exists N_2 \in N^*$ s.t. $\forall n > N_2, \forall p \in N^*$, $-\varepsilon < b_{n+1} \cdots + b_{n+p} < \varepsilon$

$\therefore$ 对 $\forall \varepsilon > 0$，取 $N = \max\{N_1, N_2\}$，则对 $\forall n > N$ 及 $\forall$ 正整数 p.

$c_{n+p} - c_n = c_{n+1} + \cdots + c_{n+p} \le b_{n+1} + \cdots + b_{n+p} < \varepsilon$

$\ge a_{n+1} + \cdots + a_{n+p} > \varepsilon$

$\therefore |c_{n+p} - c_n| < \varepsilon$

$\therefore \sum\limits_{n=1}^{\infty} c_n$ 收敛

练习14.2

1.

(1) $D_n=\frac{u_{n+1}}{u_n}=\frac{3n+2}{4n+1}\to\frac{3}{4}<1$ 收敛

(2) $D_n=\frac{u_{n+1}}{u_n}=\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n}=\frac{3}{n+1}\to 0<1$ 收敛

(3) $D_n=\frac{u_{n+1}}{u_n}=\frac{3(n+1)}{(2n+2)(2n+1)}\to 0<1$ 收敛

(4) $D_n=\frac{u_{n+1}}{u_n}=\frac{(n+1)^2}{(2n+2)(2n+1)}=\frac{n+1}{2(2n+1)}\to\frac{1}{4}<1$ 收敛

(5) $C_n=\sqrt[n]{u_n}=\frac{1}{\ln n}\to 0<1$ 收敛

(6) $C_n=\sqrt[n]{u_n}=\frac{n+1}{2n+1}\to\frac{1}{2}<1$ 收敛

(7) $C_n=\frac{\sqrt[n]{3n^3+1}}{2}$ $\lim\limits_{n\to\infty}\sqrt[n]{3n^3+1}=\lim\limits_{n\to\infty}e^{\ln(3n^3+1)^{\frac{1}{n}}}=e^{\lim\limits_{n\to\infty}\frac{\ln(3n^3+1)}{n}}=e^{\lim\limits_{n\to\infty}\frac{9n^2}{3n^3+1}}=e^0=1$

$\therefore C_n\to\frac{1}{2}<1$ 收敛

(8) $C_n=\sqrt[n]{u_n}=\sqrt[n]{n^3\cdot\frac{\sqrt{2}+(-1)^n}{3}}=(\sqrt[n]{n})^3\cdot\frac{\sqrt{2}+(-1)^n}{3}<\frac{1+\sqrt{2}}{3}\cdot(\sqrt[n]{n})^3\to\frac{1+\sqrt{2}}{3}<1$

收敛

(9) $\lim\limits_{n\to\infty}\frac{\sin^2\frac{1}{n}}{\frac{1}{n^2}}=\lim\limits_{n\to 0}\left(\frac{\sin n}{n}\right)^2=1$ 收敛

(10) $C_n=\sqrt[n]{u_n}=2\cdot\sqrt[n]{\sin\frac{x}{3^n}}=2\cdot\sqrt[n]{\frac{x}{3^n}+o\left(\frac{1}{3^n}\right)}\to\frac{2}{3}\cdot\sqrt[n]{x}$

$x>0$时 $\lim\limits_{n\to\infty}C_n=\frac{2}{3}<1$ 收敛.

$x=0$时 $\lim\limits_{n\to\infty}C_n=0<1$ 收敛

$x<0$时 $\lim\limits_{n\to\infty}C_n=-\frac{2}{3}<1$ 收敛

(11) $\lim\limits_{n\to\infty}\frac{\frac{1}{\sqrt[3]{n^2-1}}}{\frac{1}{\sqrt[3]{n^2}}}=\lim\limits_{n\to\infty}\sqrt[3]{\frac{n^2}{n^2-1}}=1$ 发散

(12) $\lim\limits_{n\to\infty}\frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{n}}=1$ 发散

(13) $\lim\limits_{n\to\infty}\frac{\frac{\ln n}{n^2}}{\frac{1}{n^{\frac{3}{2}}}}=\lim\limits_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim\limits_{n\to\infty}\frac{2}{\sqrt{n}}=0$ 收敛

(14) $\dfrac{n^{n-1}}{(2n^2+n+1)^{\frac{n-1}{2}}} = \left(\sqrt{\dfrac{n^2}{2n^2+n+1}}\right)^{n-1} < \left(\dfrac{\sqrt{2}}{2}\right)^{n-1}$

$\because \sum\left(\frac{\sqrt{2}}{2}\right)^{n-1}$ 收敛　　$\therefore$ 收敛

(15) $\lim\limits_{n\to\infty}\dfrac{\frac{1}{(\ln n)^k}}{\frac{1}{n}} = \lim\limits_{n\to\infty}\dfrac{n}{(\ln n)^k} = \lim\limits_{n\to\infty}\left(\dfrac{n^{\frac{1}{k}}}{\ln n}\right)^k$

$= \lim\limits_{n\to\infty}\left(\dfrac{\frac{1}{k}n^{\frac{1}{k}-1}}{\frac{1}{n}}\right)^k = \lim\limits_{n\to\infty}\dfrac{n}{k^k} = +\infty$

发散

(16) $\lim\limits_{n\to\infty}u_n = \lim\limits_{n\to\infty}\sqrt[n]{\dfrac{n}{(1+\frac{1}{n^2})^{n^2}}} = \lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n}}{\sqrt[n]{e}} = 1 \neq 0$

发散

(17) $\lim\limits_{n\to\infty}\dfrac{u_n}{\frac{1}{n^2}} = \lim\limits_{n\to\infty}n^2\left(\frac{1}{2}\cdot\frac{x^2}{n^2}+o\left(\frac{1}{n^2}\right)\right) = \dfrac{x^2}{2}$

收敛

(18) $\lim\limits_{n\to\infty}\dfrac{u_n}{\frac{1}{n^2}} = \lim\limits_{n\to\infty}n^2\left(\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right) = \dfrac{1}{2}$

收敛

(19) $\lim\limits_{n\to\infty}\dfrac{u_n}{\frac{1}{n^{\frac{3}{2}}}} = \lim\limits_{n\to\infty}n^{\frac{3}{2}}\cdot\dfrac{\frac{1}{n}-\ln(1+\frac{1}{n})}{\frac{1}{\sqrt{n}}+\sqrt{\ln(1+\frac{1}{n})}}$

$= \lim\limits_{n\to\infty}n^{\frac{3}{2}}\cdot\dfrac{\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)}{\frac{1}{\sqrt{n}}+\sqrt{\ln(1+\frac{1}{n})}} = 1$

收敛

(20) $\displaystyle\int_3^n\frac{1}{x\ln x\ln\ln x} = \int_3^n\frac{d\ln x}{\ln x\ln\ln x} = \int_3^n\frac{d\ln\ln x}{\ln x} = \ln\ln\ln x\Big|_3^n$

$= \ln\ln\ln n - \ln\ln\ln 3$

$\because$ 极数 $\ln\ln\ln n - \ln\ln\ln 3$ 发散

$\therefore$ 发散

(21) $\dfrac{1}{\ln(n!)} \geqslant \dfrac{1}{\ln(n^n)} = \dfrac{1}{n\ln n} \Rightarrow$ 发散

发散

练习14.3

1. $\dfrac{u_n}{u_{n+1}} = \dfrac{n+1}{|\alpha-n|}$ 当 n 充分大时，$n>\alpha$.

此时 $n\left(\dfrac{u_n}{u_{n+1}}-1\right) = \dfrac{(\alpha+1)n}{n-\alpha} = 1+\alpha+\dfrac{\alpha(\alpha+1)}{n-\alpha} \to 1+\alpha>1 \quad (n\to\infty)$

∴ 级数收敛.

2. (1) $\dfrac{u_n}{u_{n+1}} = \dfrac{a+\sqrt{n+1}}{\sqrt{n+1}} = 1+\dfrac{a}{\sqrt{n+1}}$ $\quad n\left(\dfrac{u_n}{u_{n+1}}-1\right) = \dfrac{an}{\sqrt{n+1}} = a\cdot\sqrt{\dfrac{n^2}{n+1}} \to +\infty > 1$

∴ 级数收敛

(2) $\dfrac{u_n}{u_{n+1}} = \dfrac{q+n+1}{n+1}\cdot\left(\dfrac{n}{n+1}\right)^{-p}$

$n\left(\dfrac{u_n}{u_{n+1}}-1\right) = (q+n+1)\cdot\left(\dfrac{n}{n+1}\right)^{1-p} - n$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\left[(q+n+1)\cdot\left(1-\frac{1-p}{n+1}+o\left(\frac{1}{n}\right)\right)-n\right]$$

$$= \lim_{n\to\infty}\left[q+n+1+\frac{(p-1)(q+n+1)}{n+1}-n\right]$$

$$= \lim_{n\to\infty}\left[q+1+(p-1)\cdot\left(1+\frac{q}{n+1}\right)\right]$$

$$= p+q$$

∴ $p+q>1$ 时 $\Rightarrow$ 级数收敛

$p+q<1$ 时 $\Rightarrow$ 级数发散

练习14.4

1.

(1) 令 $a_k=\frac{1}{(2k-1)!}$，$b_k=\frac{1}{2k}$

$\because \lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{1}{2n(2n+1)}=0<1$ $\therefore$ 级数 $\sum a_k$ 收敛

$\because$ 级数 $\sum\frac{1}{n}$ 发散，且 $\sum b_k=\frac{1}{2}\sum\frac{1}{k}$ $\therefore$ 级数 $\sum b_k$ 发散

$\therefore$ 原级数 $=\sum a_k-\sum b_k$ 发散.

(2) 令 $f(x)=\frac{\ln x}{\sqrt{x}}$ $f'(x)=\frac{\frac{1}{\sqrt{x}}-\frac{\ln x}{2\sqrt{x}}}{x}=\frac{2-\ln x}{2x\sqrt{x}}$

$\therefore$ 在 $n\geq 8$ 时，$\left\{\frac{\ln n}{\sqrt{n}}\right\}$ 递减. 又 $\lim\limits_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim\limits_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}=\lim\limits_{n\to\infty}\frac{2}{\sqrt{n}}=0$

$\therefore$ 级数 $\sum(-1)^{n+1}\frac{\ln n}{\sqrt{n}}$ 收敛

$\Rightarrow$ 对于级数 $\sum\frac{\ln n}{\sqrt{n}}$

$\because \frac{\frac{\ln n}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}=\ln n\to+\infty$ $\therefore$ 级数 $\sum\frac{\ln n}{\sqrt{n}}$ 发散

故原级数条件收敛

(3) 当 $n>\left[\frac{2x}{\pi}\right]$ 时. $\left\{\sin\frac{x}{n}\right\}$ 递减且收敛于零.

$\therefore (-1)^n\sin\frac{x}{n}$ 收敛. 又 $\{\arctan n\}$ 单调有界

$\therefore$ 级数 $\sum(-1)^n\sin\frac{x}{n}\arctan n$ 收敛

$\Rightarrow$ 对于级数 $\sum\sin\frac{x}{n}\arctan n$

$\because \lim\limits_{n\to\infty}\frac{\sin\frac{x}{n}\arctan n}{\frac{1}{n}}=\lim\limits_{n\to\infty}\frac{x\cdot\sin\frac{x}{n}}{\frac{x}{n}}\cdot\lim\limits_{n\to\infty}\arctan n$

$=\frac{\pi}{2}x$ $\therefore$ 此级数发散

故原级数条件收敛

(4) $\sin(\sqrt{n^2+1}\pi) = \sin[(\sqrt{n^2+1}-n)\pi + n\pi]$

$= \sin(\sqrt{n^2+1}-n)\pi \cos n\pi = \cos n\pi \cdot \sin\frac{\pi}{\sqrt{n^2+1}+n}$

显然 $\left\{\sin\frac{\pi}{\sqrt{n^2+1}+n}\right\}$ 单调趋于零，

又 $\sum_{n=1}^{m}\cos n\pi \in \{0,-1\}$ 即 $\{\cos n\pi\}$ 部分和有界

∴ 级数收敛

⇒ 对于级数 $\sum\left|\sin(\sqrt{n^2+1}\pi)\right|$

∵ $\sin(\sqrt{n^2+1}\pi) = \cos n\pi \sin\frac{\pi}{\sqrt{n^2+1}+n} = \pm\sin\frac{\pi}{\sqrt{n^2+1}+n}$ 且 $\frac{\pi}{\sqrt{n^2+1}+n} < \frac{\pi}{2}$

∴ $\left|\sin(\sqrt{n^2+1}\pi)\right| = \sin\frac{\pi}{\sqrt{n^2+1}+n}$

$$\lim_{n\to\infty}\frac{\sin\frac{\pi}{\sqrt{n^2+1}+n}}{\frac{1}{n}} = \lim_{n\to\infty} n\sin\frac{\pi}{\sqrt{n^2+1}+n}$$

$$= \lim_{n\to\infty}\frac{n\pi}{\sqrt{n^2+1}+n} = \lim_{n\to\infty}\frac{\pi}{1+\sqrt{1+\frac{1}{n^2}}} = \frac{\pi}{2}$$

∴ 级数 $\sum\sin\frac{\pi}{\sqrt{n^2+1}+n}$ 发散

故原级数条件收敛

(5) 记 $u_n = \frac{(2n-1)!!}{(2n)!!}$

$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} < 1$ ∴ $\{u_n\}$ 递减

$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\sqrt{u_n^2}$, 而 $u_n^2 = \frac{(2n-1)!!\cdot(2n-1)!!}{4\cdot(n!)^2} < \frac{1}{4\cdot n!}$

∴ $u_n \to 0$

∴ 级数 $\sum(-1)^{n+1}u_n$ 收敛

⇒ 对于级数 $\sum u_n$

∵ $n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n}{2n+1} \to \frac{1}{2} < 1$ 级数发散

∴ 原级数条件收敛

(6) 由于数列 $\left\{(-1)^{\frac{n(n+1)}{2}}\right\}$ 部分和有界，数列 $\left\{\frac{1}{\sqrt{n}}\right\}$ 单调趋于零

∴ 级数收敛

⇒ 对于级数 $\sum \frac{1}{\sqrt{n}}$ 显然发散

∴ 原级数条件收敛

(7) 由于 $\sum_{k=1}^{24} \sin \frac{k}{12}\pi = 0$ 　记 $h = m - 24\left[\frac{m}{24}\right] + 1$

∴ $\sum_{n=1}^{m} \sin \frac{n\pi}{12} = \sum_{n=h}^{m} \sin \frac{n\pi}{12} < h - m + 1$

由于数列 $\left\{\sin \frac{n\pi}{12}\right\}$ 部分和有界，数列 $\left\{\frac{1}{\ln n}\right\}$ 单调趋于零 　∴ 级数收敛

⇒ 对于级数 $\sum \left|\frac{\sin \frac{n}{12}\pi}{\ln n}\right|$ 由周期性可知其发散性：

取一子列 $\{n_k\}$ 满足 $n_k \in (24k+3, 24k+9)$ 　$k = 0, 1, 2, \cdots$

此时 $\sin \frac{n_k \pi}{12}$ 满足 $\sin \frac{n_k}{12}\pi > \frac{\sqrt{2}}{2}$

∴ $\sum \left|\frac{\sin \frac{n}{12}\pi}{\ln n}\right| > \sum \left|\frac{\sin \frac{n_k}{12}\pi}{\ln n}\right| = \sum \frac{\sin \frac{n_k}{12}\pi}{\ln n} \geqslant \sum \frac{\frac{\sqrt{2}}{2}}{\ln k} \to \infty$ 发散 　∴ $\sum \left|\frac{\sin \frac{n}{12}\pi}{\ln n}\right|$ 发散

综上，原级数条件收敛

(8) 由于 $(-1)^{n-1} \sin n = \sin(n\pi - n) = \sin n(\pi - 1)$

令 $b_n = \sin n(\pi - 1)$，$a_n = \frac{1}{n}$

显然 $B_n = \sum b_n$ 有界，a_n 单调趋于零 　原级数收敛

模仿⑤中做法，可证 $\sum \left|\frac{\sin n}{n}\right|$ 发散． 　条件收敛

练习14.5

1. $\lim\limits_{n\to\infty}\left(\dfrac{\sum\limits_{k=1}^{n}a_k^{+}}{\sum\limits_{k=1}^{n}a_k^{-}}-1\right) = \lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}a_k^{+}-\sum\limits_{k=1}^{n}a_k^{-}}{\sum\limits_{k=1}^{n}a_k^{-}} = \lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}a_k}{\sum\limits_{k=1}^{n}a_k^{-}}$

$\because$ $\sum a_n$ 条件收敛　即 $\{a_n\}$ 收敛 且 $\{a_k^{-}\}$ 发散

$\therefore$ $\sum\limits_{k=1}^{n}a_k = C$, $\sum\limits_{k=1}^{n}a_k^{-} = +\infty$

$\therefore$ 原式$=0$　也即 $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}a_k^{+}}{\sum\limits_{k=1}^{n}a_k^{-}}=1$

练习14.6

1\. (1) 同例3 p45~46

(2) $P_n=\prod_{k=1}^{n}(1+x^{2^{k-1}})$

$=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})$

$=\dfrac{(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})}{1-x}$

$=\dfrac{(1-x^2)(1+x^2)(1+x^4)\cdots(1+x^{2^{n-1}})}{1-x}$

$=\cdots=\dfrac{1-x^{2^n}}{1-x}$ 　又 $|x|<1$，$\therefore \lim\limits_{n\to\infty}P_n=\dfrac{1}{1-x}$

$\therefore \prod_{n=1}^{\infty}(1+x^{2^{n-1}})=\dfrac{1}{1-x}$

2\.

(1) $P_n=\dfrac{2\times\cancel{2}}{1\times\cancel{3}}\times\dfrac{\cancel{3}\times\cancel{3}}{\cancel{2}\times\cancel{4}}\times\dfrac{\cancel{4}\times\cancel{4}}{\cancel{3}\times\cancel{5}}\times\dfrac{5\times5}{4\times6}\cdots\dfrac{\cancel{n^2}}{\cancel{(n-1)(n+1)}}\cdot\dfrac{(n+1)^2}{n(n+2)}$

$=\dfrac{2(n+1)}{n+2}=2(1-\dfrac{1}{n+2})$ 　$\lim\limits_{n\to\infty}P_n=2$ 　$\therefore \prod_{n=1}^{\infty}\dfrac{(n+1)^2}{n(n+2)}$ 收敛

(2) 令 $u_n=\sqrt[n]{1+\frac{1}{n}}$ 　$\ln u_n=\dfrac{\ln(1+\frac{1}{n})}{n}$

$\because \dfrac{\ln u_n}{\frac{1}{n^2}}=n\ln(1+\frac{1}{n})=n(\frac{1}{n}+o(\frac{1}{n}))=1+o(1)\to1 \quad (n\to\infty)$

$\therefore$ 级数 $\sum\ln u_n$ 收敛

$\therefore \prod_{n=1}^{\infty}u_n$ 收敛

(3) 令 $u_n=\dfrac{n}{\sqrt{n^2+1}}$ 　$\ln u_n=\dfrac{1}{2}\ln\dfrac{n^2}{n^2+1}=\dfrac{1}{2}\ln(1-\dfrac{1}{n^2+1})$

$\therefore \dfrac{\ln\frac{n^2}{n^2+1}}{\frac{1}{n^2}}=n^2(-\dfrac{1}{n^2+1}-\dfrac{1}{2(n^2+1)^2}+o(\dfrac{1}{n^4}))$

$=-\dfrac{n^2}{n^2+1}+o(1)\to-1 \quad (n\to\infty)$

$\therefore$ 级数 $\sum\ln u_n$ 收敛 　$\therefore \prod_{n=1}^{\infty}u_n$ 收敛

(4) 令 $u_n=\sqrt[n]{\ln(n+x)-\ln n}$

$$\ln u_n = n\ln\left(\ln\left(1+\frac{x}{n}\right)\right)$$

$$\lim_{n\to\infty}\frac{\ln u_n}{\frac{1}{n}} = \lim_{n\to\infty}\ln\left(\ln\left(1+\frac{x}{n}\right)\right) = \lim_{n\to\infty}\ln\frac{x}{n} = -\infty$$

∴ 级数 $\sum -\ln u_n$ 收敛　∴ 级数 $\sum \ln u_n$ 收敛

即 $\prod\limits_{n=1}^{\infty} u_n$ 收敛.

练习14.7

1. 记数列$\{a_n\}$: $1, q, q^2, \cdots$

$\{b_n\}$: $1, q, q^2, \cdots$

则 $\sum_{n=1}^{\infty}(a_1b_n+a_2b_{n-1}+\cdots+a_nb_1) = 1+2q+3q^2+4q^3+\cdots$

$= \sum_{n=0}^{\infty}(n+1)q^n$

由柯西定理，上式 $= \sum_{n=1}^{\infty}a_n \cdot \sum_{n=1}^{\infty}b_n = \left(\sum_{n=0}^{\infty}q^n\right)^2$

$$\begin{array}{lllll}
1 & q & q^2 & q^3 & \cdots \\
q & q^2 & q^3 & q^4 & \cdots \\
q^2 & q^3 & q^4 & q^5 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

2. (1) 记 $a_n = (-1)^n\frac{x^{2n+1}}{(2n+1)!}$， $\{a_n\}$: $x, -\frac{x^3}{3!}, \frac{x^5}{5!}, \cdots$

记 $b_n = (-1)^n\frac{x^{2n}}{(2n)!}$， $\{b_n\}$: $1, -\frac{x^2}{2!}, \frac{x^4}{4!}, \cdots$

对于 $\sum_{k=0}^{\infty}(a_0b_k+a_1b_{k-1}+\cdots+a_kb_0)$,

记 $c_n = \frac{1}{(2n+1)!}$， $d_n = \frac{1}{(2n)!}$ $(n=0,1,2,\cdots)$

∴ 记 $V_k = c_0d_k+c_1d_{k-1}+\cdots+c_kd_0$

$= \sum_{m=0}^{k}\frac{1}{m!(2k+1-m)!}$

$= \frac{1}{(2k+1)!}\sum_{m=0}^{k}\frac{(2k+1)!}{m!(2k+1-m)!} = \frac{1}{(2k+1)!}\cdot\sum_{m=0}^{k}C_{2k+1}^{m} = \frac{1}{(2k+1)!}\cdot\frac{1}{2}\cdot\sum_{m=0}^{2k+1}C_{2k+1}^{m}$

$= \frac{1}{2}\cdot\frac{1}{(2k+1)!}\cdot 2^{2k+1} = \frac{4^k}{(2k+1)!}$

由于 $a_0b_k+a_1b_{k-1}+\cdots+a_kb_0 = (-1)^k\cdot V_k\cdot x^{2k+1} = (-1)^k\cdot\frac{4^k\cdot x^{2k+1}}{(2k+1)!}$

$\therefore \sum_{k=0}^{\infty}(a_0b_k+a_1b_{k-1}+\cdots+a_kb_0) = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\cdot\frac{(2x)^{2k+1}}{(2k+1)!} = \frac{1}{2}S(2x)$

又由柯西定理，

$\sum_{k=0}^{\infty}(a_0b_k+a_1b_{k-1}+\cdots+a_kb_0) = \sum_{k=0}^{\infty}a_k\cdot\sum_{k=0}^{\infty}b_k = S(x)C(x)$

$\therefore S(2x) = 2S(x)C(x)$

$$\begin{array}{lll}
a_0b_0 & a_0b_1 & a_0b_2 \\
a_1b_0 & b_1b_1 & \\
a_2b_0 & &
\end{array}$$

$$\begin{array}{llll}
x & -\frac{x^3}{2!} & \frac{x^5}{4!} & -\frac{x^7}{6!} \\
-\frac{x^3}{3!} & \frac{x^5}{3!\cdot 2!} & -\frac{x^7}{3!\cdot 4!} & \\
\frac{x^5}{5!} & \frac{-x^7}{5!\cdot 2!} & & \\
-\frac{x^7}{7!} & & &
\end{array}$$

(2) $S^2(x) = \sum_{n=0}^{\infty} (a_0 a_n + a_1 a_{n-1} + \cdots + a_n a_0)$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k a_{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n x^{2n+2}}{(2k+1)!\,(2n-2k+1)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n x^{2n+2}}{(2k+1)!\,(2n-2k+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \left(\sum_{k=0}^{n} \frac{(2n+2)!}{(2k+1)!\,(2n-2k+1)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \left(\sum_{k=0}^{n} C_{2n+2}^{2k+1} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \left(C_{2n+2}^{1} + C_{2n+2}^{3} + C_{2n+2}^{5} + \cdots \right)$$

$C^2(x) = \sum_{n=0}^{\infty} (b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0)$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_k b_{n-k} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(-1)^n x^{2n}}{(2k)!\,(2n-2k)!} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \left(\sum_{k=0}^{n} \frac{(2n)!}{(2k)!\,(2n-2k)!} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \left(\sum_{k=0}^{n} C_{2n}^{2k} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \left(C_{2n}^{0} + C_{2n}^{2} + C_{2n}^{4} + \cdots \right)$$

$$= 1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \left(C_{2n+2}^{0} + C_{2n+2}^{2} + C_{2n+2}^{4} + \cdots \right)$$

$\because C_{2n+2}^{1} + C_{2n+2}^{3} + C_{2n+2}^{5} + \cdots C_{2n+2}^{2n+1} = C_{2n+2}^{0} + \cdots + C_{2n+2}^{2n+2}$

$\therefore S^2(x) + C^2(x)$

$$= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \left(C_{2n+2}^{1} + C_{2n+2}^{3} + C_{2n+2}^{5} + \cdots \right) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \left(C_{2n+2}^{0} + C_{2n+2}^{2} + C_{2n+2}^{4} + \cdots \right)$$

$= 1$

习题14(A).

1、证明:

(法一). $\because \sum a_n$ 收敛 $\therefore \lim\limits_{n\to\infty} a_n = 0$

又 $\{a_n\}$ 递减 $\therefore$ 对 $\forall n$, $a_n \geq 0$

由柯西收敛原理

对 $\forall \varepsilon>0$ $\exists N\in\mathbb{N}^*$. $\forall n>N$, $p\in\mathbb{N}^*$, $\left|\sum\limits_{k=n+1}^{n+p} a_k\right|<\varepsilon$

特别地，取 $p=n$, 则有

$$0<\frac{1}{2}\cdot(2n)a_{2n} = a_{2n}+a_{2n}+\cdots+a_{2n}$$
$$<a_{n+1}+a_{n+2}+\cdots+a_{n+n}<\varepsilon$$

即 $\lim\limits_{n\to\infty} 2n a_{2n} = 0$

又由 $\{a_n\}$ 递减，故 $\lim\limits_{n\to\infty} 2n a_{2n+1} = 0$ 从而 $\lim\limits_{n\to\infty}(2n+1)a_{2n+1} = \lim\limits_{n\to\infty}\frac{2n+1}{2n}\cdot 2n a_{2n+1} = 0$

$$\therefore \lim_{n\to\infty} n a_n = \begin{cases} \lim\limits_{n\to\infty} 2k a_{2k} & n\text{为偶} \\ \lim\limits_{n\to\infty}(2k+1)a_{2k+1} & n\text{为奇} \end{cases} = 0$$

(法二) 反设 $\lim\limits_{n\to\infty} n a_n = l \neq 0$, 则 $l\in(0,+\infty]$

$\{a_n\}$ 是 $\{\frac{1}{n}\}$ 低阶无穷小, $\therefore \sum a_n$ 发散

矛盾.

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2. (1) $\because \lim\limits_{n\to\infty}\frac{1}{n^p-n^q}=\lim\limits_{n\to\infty}\frac{1}{n^p}\cdot\frac{1}{1-n^{q-p}}=\lim\limits_{n\to\infty}\frac{1}{n^p}$

$\therefore \frac{1}{n^p-n^q}\sim\frac{1}{n^p}$

$\therefore$ $0<p\le1$ 时，$\sum\frac{1}{n^p-n^q}$ 发散

$p>1$ 时．$\sum\frac{1}{n^p-n^q}$ 收敛

(2) $\lim\limits_{n\to\infty}\frac{1}{p^n-q^n}=\lim\limits_{n\to\infty}\frac{1}{p^n}\cdot\frac{1}{1-\left(\frac{q}{p}\right)^n}=\lim\limits_{n\to\infty}\frac{1}{p^n}$

$\therefore \frac{1}{p^n-q^n}\sim\frac{1}{p^n}$

$\therefore$ $0<p\le1$ 时 $\sum\frac{1}{p^n-q^n}$ 发散

$p>1$ 时 $\sum\frac{1}{n^p-n^q}$ 收敛

(3) $\because \sqrt[n]{u_n}=\frac{x}{(\sqrt[n]{n})^s}$ $\lim\limits_{n\to\infty}\sqrt[n]{u_n}=x$

$\therefore$ $0<x<1$ 时，$\sum\frac{x^n}{n^s}$ 收敛

$x>1$ 时．$\sum\frac{x^n}{n^s}$ 发散

$x=1$ 时．$\begin{cases} s>1 \text{时}, & \sum\frac{x^n}{n^s} \text{收敛} \\ 0<s\le1 \text{时} & \sum\frac{x^n}{x^s} \text{发散} \end{cases}$

2. (4)

① 当 $p=1$ 时.

1° 若 $\boxed{\delta=1}$, $u_n=\dfrac{1}{n\ln n(\ln\ln n)}$. $\because \displaystyle\int_3^n \frac{dx}{x\ln x(\ln\ln x)}=\ln\ln\ln n-\ln\ln\ln 3$ 发散

$\therefore \sum u_n$ 发散

2° 若 $\boxed{0<\delta<1}$ 则 $\dfrac{1}{n(\ln n)^p(\ln\ln n)^\delta}>\dfrac{1}{n(\ln n)\cdot(\ln\ln n)}$ 由 $\sum\dfrac{1}{n\ln n\ln\ln n}$ 的发散

$\therefore \sum u_n$ 发散

3° 若 $\boxed{\delta>1}$ 则 $\displaystyle\int_3^n \frac{dx}{x\ln x(\ln\ln x)^\delta}=\frac{(\ln\ln n)^{1-\delta}}{1-\delta}-\frac{(\ln\ln 3)^{1-\delta}}{1-\delta}$ 收敛 $\therefore \sum u_n$ 收敛

② 当 $0<p<1$ 时.

1° 若 $\boxed{0<\delta\le 1}$, 则 $u_n\ge\dfrac{1}{n\ln n(\ln\ln n)}$ $\therefore \sum u_n$ 发散

2° 若 $\boxed{\delta>1}$ 由于 $\displaystyle\lim_{n\to\infty}\frac{n\ln n(\ln\ln n)}{n(\ln n)^p(\ln\ln n)^\delta}=\lim_{n\to\infty}\frac{(\ln n)^{1-p}}{(\ln\ln n)^{\delta-1}}=+\infty$ (x 发散速度快于 $\ln x$, $1-p>0$, $\delta-1>0$)

$\therefore \dfrac{1}{n\ln n(\ln\ln n)}$ 是 $\dfrac{1}{n(\ln n)^p(\ln\ln n)^\delta}$ 的高阶无穷小 $\therefore \sum u_n$ 发散

③ 当 $p>1$ 时.

由于 $\displaystyle\lim_{n\to\infty}\frac{n(\ln n)^p(\ln\ln n)^\delta}{n\cdot(\ln n)^{\frac{p+1}{2}}}=\lim_{n\to\infty}(\ln n)^{\frac{p-1}{2}}\cdot(\ln\ln n)^\delta=+\infty$ ($\delta\ge1$ 时. 显然. $0<\delta<1$ 时, $\ln x$ 发散速度慢于 x)

$\therefore \dfrac{1}{n(\ln n)^p(\ln\ln n)^\delta}$ 是 $\dfrac{1}{n\cdot(\ln n)^{\frac{p+1}{2}}}$ 的高阶无穷小, 又 $\dfrac{1}{n(\ln n)^{\frac{p+1}{2}}}$ 中, $\dfrac{p+1}{2}>1$.

(由 P14 例 6) $\sum\dfrac{1}{n(\ln n)^{\frac{p+1}{2}}}$ 收敛 $\therefore \sum\dfrac{1}{n(\ln n)^p(\ln\ln n)^\delta}$ 收敛 $\sum u_n$ 收敛

综上:

$0<p\le1$ 且 $0<\delta\le1$ 时. 级数发散.

在 $p>1$ 或 $\delta>1$ 时. 级数收敛.

3、证明:

当 $\sum a_n$ 收敛时. $\forall \varepsilon>0$. $\exists N_0$, $\forall n>N_0$. $\forall p$. $|a_{n+1}+\cdots+a_{n+p}|<\varepsilon$.

由于 $\left|\sum_{k=n+1}^{n+p} 2^k a_{2^k}\right| = \left|2^{n+1}a_{2^{n+1}} + 2^{n+2}a_{2^{n+2}} + \cdots + 2^{n+p}a_{2^{n+p}}\right|$

$< 2\left|(a_{2^n}+a_{2^n+1}+\cdots+a_{2^{n+1}-1})+(a_{2^{n+1}}+a_{2^{n+1}+1}+\cdots+a_{2^{n+2}-1})+\cdots+(a_{2^{n+p-1}}+a_{2^{n+p-1}+1}+\cdots+a_{2^{n+p}-1})\right|$

$= 2\left|\sum_{k=2^n}^{2^{n+p}-1} a_k\right| < 2\left|\sum_{k=2^n}^{2^{n+p}} a_k\right|$

设 $2^{n+p}=2^n+p_0$. $\therefore 2^n>n>N_0$ 时, $\left|\sum_{k=2^n}^{2^n+p_0} a_k\right|<\varepsilon$ $\therefore$ 上式 $<2\varepsilon$

$\therefore \sum 2^k a_{2^k}$ 收敛.

当 $\sum a_n$ 发散时. $\forall N\in\mathbb{N}^*$, $\exists \varepsilon_0>0$. $\forall n>N_0$, $\forall p$ $|a_{n+1}+\cdots+a_{n+p}|\geq\varepsilon_0$.

由于 $\left|\sum_{k=n+1}^{n+p} 2^k a_{2^k}\right| = \left|2^{n+1}a_{2^{n+1}} + 2^{n+2}a_{2^{n+2}} + \cdots + 2^{n+p}a_{2^{n+p}}\right|$

$> \left|(a_{2^{n+1}}+a_{2^{n+1}+1}+\cdots+a_{2^{n+2}-1})+(a_{2^{n+2}}+a_{2^{n+2}+1}+\cdots+a_{2^{n+3}-1})+\cdots+(a_{2^{n+p}}-a_{2^{n+p}+1}+\cdots+a_{2^{n+p+1}-1})\right|$

$= \left|\sum_{k=2^{n+1}}^{2^{n+p+1}-1} a_k\right| \doteq \left|\sum_{2^{n+1}}^{2^{n+1}+p_0} a_k\right| \geq \varepsilon_0 > 0$

$\therefore \sum 2^k a_{2^k}$ 发散

综上: $\sum a_n$ 与 $\sum 2^n a_{2^n}$ 同敛散

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4. 证:

$\{a_n\}$有界时. $\because\{a_n\}$是递增的正项数列. 由单调有界定理$\{a_n\}$收敛. 设$\lim_{n\to\infty}a_n=A$, $A>0$

$\therefore \lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1$ 即$\exists N\in\mathbb{N}^*$, $n>N$时, 对$\forall\varepsilon>0$, $\left|\frac{a_n}{a_{n+1}}-1\right|<\varepsilon$

$\therefore \left|\sum_{n=k+1}^{k+p}\left(1-\frac{a_n}{a_{n+1}}\right)\right|\le\sum_{n=k+1}^{k+p}\left|1-\frac{a_n}{a_{n+1}}\right|=\left|1-\frac{a_{k+1}}{a_{k+2}}\right|+\left|1-\frac{a_{k+2}}{a_{k+3}}\right|+\cdots+\left|1-\frac{a_{k+p}}{a_{k+p+1}}\right|<p\varepsilon$.

当$k>N$时. 由柯西收敛定理, $\sum\left(1-\frac{a_n}{a_{n+1}}\right)$收敛.

$\{a_n\}$无界时. 假设$\sum\left(1-\frac{a_n}{a_{n+1}}\right)$收敛. $\because\lim_{n\to\infty}a_n=+\infty$. $\therefore\lim_{n\to\infty}\frac{1}{a_n}=0$ $\therefore 0<\frac{1}{a_n}\le\frac{1}{a_1}$

$\therefore\left\{\frac{1}{a_n}\right\}$为单调有界数列.

由阿贝尔判别法. $\sum\frac{1}{a_n}\left(1-\frac{a_n}{a_{n+1}}\right)=\sum\left(\frac{1}{a_n}-\frac{1}{a_{n+1}}\right)$是收敛级数.

$\therefore S_n=\frac{1}{a_1}-\frac{1}{a_{n+1}}$ 有界, 设$\left|\frac{1}{a_1}-\frac{1}{a_{n+1}}\right|\le M$ $\therefore|a_{n+1}|\le\frac{1}{\left|\frac{1}{a_1}\right|-M}$.

与$\{a_n\}$无界矛盾 $\therefore\sum\left(1-\frac{a_n}{a_{n+1}}\right)$发散.

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5.

(1) 由taylor展开，$\lim_{n\to\infty}\frac{n^2}{(\ln n)^{\ln n}}=0$ 即$\frac{1}{(\ln n)^{\ln n}}$是$\frac{1}{n^2}$的高阶无穷小、

$\because \sum\frac{1}{n^2}$收敛 由阶估计，$\sum\frac{1}{(\ln n)^{\ln n}}$也收敛

(2) 由taylor展开 $\lim_{n\to\infty}\frac{(\ln n)^{\ln\ln n}}{n}=0$ 即$\frac{1}{n}$是$\frac{1}{(\ln n)^{\ln\ln n}}$的高阶无穷小

$\because \sum\frac{1}{n}$发散 由阶估计 $\sum\frac{1}{(\ln n)^{\ln\ln n}}$也发散

(3) 由于 $\lim_{n\to\infty}\frac{n^2}{(\ln\ln n)^{\ln n}}=0$ 即$\frac{1}{(\ln\ln n)^{\ln n}}<\frac{1}{n^2}$ $\sum\frac{1}{n^2}$收敛$\Rightarrow\sum\frac{1}{(\ln\ln n)^{\ln n}}$收敛.

注：

(1)中，$(\ln n)^{\ln n}=(e^{\ln\ln n})^{\ln n}=(e^{\ln n})^{\ln\ln n}=n^{\ln\ln n}>n^2 \quad (n\to\infty)$

$\therefore \frac{1}{n^{\ln\ln n}}<\frac{1}{n^2}$

(2)中. $(\ln n)^{\ln\ln n}=(e^{\ln\ln n})^{\ln\ln n}=e^{\ln^2\ln n}<e^{\ln n}=n$

$\therefore \frac{1}{(\ln n)^{\ln\ln n}}>\frac{1}{n}$

(3)中. $(\ln\ln n)^{\ln n}=(e^{\ln\ln\ln n})^{\ln n}=(e^{\ln n})^{\ln\ln\ln n}=n^{\ln\ln\ln n}>n^2 \quad (n\to\infty)$

$\therefore \frac{1}{(\ln\ln n)^{\ln n}}<\frac{1}{n^2}$

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6、∵ 正项级数 $\sum a_n$ 收敛，

∴ $\lim\limits_{n\to\infty} a_n=0$ ∴对 $\forall\varepsilon>0$，$\exists N\in\mathbb{N}^*$，$n>N$ 时，$|a_n|<\varepsilon$. 取 $\varepsilon=1$

设 $N_1\in\mathbb{N}^*$. $n>N_1$ 时 $|a_n|<1$.

由 $\sum a_n$ 收敛，$\exists N_2$，$n>N_2$ 时，对 $\forall\varepsilon>0$，$n>N_2$，$\forall p\in\mathbb{N}^*$.

$|a_{n+1}+\cdots+a_{n+p}|<\varepsilon$.

取 $N=\max\{N_1,N_2\}$

$n>N$ 时. $\forall |a_n|<1$ ∴ 对 $r>1$，$|a_n|>|a_n|^r$. $a_n>0$，$a_n>a_n^r$

∴ $|a_{n+1}^r+\cdots\cdots+a_{n+p}^r|<|a_{n+1}+\cdots\cdots+a_{n+p}|<\varepsilon$.

由柯西收敛原理，$\sum a_n^r$ 收敛

另：逆命题为假，如 $a_n=\frac{1}{n}$ 时. 不符合.

7.

$\sum\frac{u_k}{S_k}$ 收敛性：$\left|\sum\limits_{k=n+1}^{n+m}\frac{u_k}{S_k}\right|=\frac{u_{n+1}}{S_{n+1}}+\cdots+\frac{u_{n+m}}{S_{n+m}}>\frac{u_{n+1}+\cdots+u_{n+m}}{S_{n+m}}$ (由 $\{S_n\}$ 递增) $=\frac{S_{n+m}-S_n}{S_{n+m}}=1-\frac{S_n}{S_{n+m}}$

由 $u_n>0$，$\sum u_n$ 发散 ∴ 对任意给定的 n，可取到合适的 m 使 $\frac{S_n}{S_{n+m}}<\frac{1}{2}$

∵ $\left|\sum\limits_{k=n+1}^{n+m}\frac{u_k}{S_k}\right|=1-\frac{S_n}{S_{n+m}}>\frac{1}{2}$ ∴ $\sum\frac{u_k}{S_k}$ 发散

$\sum\frac{u_k}{S_k^2}$ 收敛性：$\frac{u_n}{S_n^2}=\frac{S_n-S_{n-1}}{S_n^2}<\frac{S_n-S_{n-1}}{S_nS_{n-1}}=\frac{1}{S_{n-1}}-\frac{1}{S_n}$

而 $\sum\limits_{k=2}^{n}\frac{u_k}{S_k^2}=\sum\limits_{k=2}^{n}\left(\frac{1}{S_{k-1}}-\frac{1}{S_k}\right)=\frac{1}{S_1}-\frac{1}{S_n}\to\frac{1}{S_1}$ ∴ $\sum\frac{u_k}{S_k^2}$ 收敛.

$\sum\frac{u_k}{S_k^{1+\sigma}}$ 收敛性：$\frac{u_n}{S_n^{1+\sigma}}=\frac{S_n-S_{n-1}}{S_n^{1+\sigma}}=\int_{S_{n-1}}^{S_n}\frac{dx}{S_n^{1+\sigma}}<\int_{S_{n-1}}^{S_n}\frac{dx}{x^{1+\sigma}}=\frac{1}{\sigma}\left(\frac{1}{S_{n-1}^{\sigma}}-\frac{1}{S_n^{\sigma}}\right)$

$(x\in(S_{n-1},S_n)$，$x<S_n)$

∴ $\sum\frac{1}{\sigma}\left(\frac{1}{S_{n-1}^{\sigma}}-\frac{1}{S_n^{\sigma}}\right)$ 收敛 ∴ $\sum\frac{u_n}{S_n^{1+\sigma}}$ 收敛

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8. (1)

• 之于级数 $\sum \frac{\cos nx}{n^p}$：

由于 $\sum_{k=1}^{n} \cos kx = \cos x + \cos 2x + \cdots + \cos nx$

$$= \frac{2\sin\frac{x}{2}}{2\sin\frac{x}{2}}(\cos x + \cos 2x + \cdots + \cos nx) = \frac{1}{2\sin\frac{x}{2}}\left(2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + \cdots\cdots\right)$$

$$= \frac{1}{2\sin\frac{x}{2}}\left(\sin\frac{3}{2}x - \sin\frac{x}{2} + \sin\frac{5}{2}x - \sin\frac{3}{2}x + \cdots + \sin\left(\frac{x}{2}+nx\right) - \sin\left(nx - \frac{x}{2}\right)\right)$$

$$= \frac{1}{2\sin\frac{x}{2}}\left(\sin\left(\frac{x}{2}+nx\right) - \sin\frac{x}{2}\right) \leq \frac{1}{2\sin\frac{x}{2}} - \frac{1}{2}$$

$\because \{\cos nx\}$ 部分和有界，$\{\frac{1}{n^p}\}$ 单调趋于零 $\therefore \sum \frac{\cos nx}{n^p}$ 收敛.

• 之于级数 $\sum \left|\frac{\cos nx}{n^p}\right|$：

① $p>1$ 时，由于 $\left|\frac{\cos nx}{n^p}\right| \leq \frac{1}{n^p}$，$\sum \frac{1}{n^p}$ 收敛 $\therefore \sum \left|\frac{\cos nx}{n^p}\right|$ 收敛.

② $0<p<1$ 时. $\because |\cos nx| \leq 1$ $\therefore \left|\frac{\cos nx}{n^p}\right| \geq \frac{\cos^2 nx}{n^p} = \frac{\cos 2nx + 1}{2n^p}$

由于 $\sum_{n=1}^{m} \cos 2nx$ 有界而 $\{\frac{1}{2n^p}\}$ 单调趋于零 故 $\sum \frac{\cos 2nx}{2n^p}$ 收敛

又 $\sum \frac{1}{2n^p}$ 发散 $\therefore \sum \frac{\cos 2nx + 1}{2n^p}$ 发散 $\therefore \sum \left|\frac{\cos nx}{n^p}\right|$ 发散

综上：

$p>1$ 时. $\sum \frac{\cos nx}{n^p}$ 绝对收敛

$0<p<1$ 时 $\sum \frac{\cos nx}{n^p}$ 条件收敛

8. (2) 记 $u_n=(-1)^n a_n$

$\therefore \sum_{n=1}^{\infty} u_n=\sum_{k=1}^{\infty}\frac{1}{(2k)^q}-\sum_{k=1}^{\infty}\frac{1}{(2k-1)^p}$. 记 $M_k=\frac{1}{(2k)^q}$, $N_k=\frac{1}{(2k-1)^p}$

① $p>1$ 且 $q>1$ 时. $\because \sum M_k, \sum N_k$ 都收敛, $\therefore \sum u_n=\sum M_k-\sum N_k$ 收敛, $\sum|u_n|=\sum M_k+\sum N_k$ 收敛

$\therefore \{u_n\}$ 绝对收敛.

② $0<p\le 1$ 且 $q>1$ 时. $\because \sum M_k$ 发散, $\sum N_k$ 收敛. $\therefore \sum u_n=\sum M_k-\sum N_k$ 发散, $\sum|u_n|=\sum M_k+\sum N_k$ 发散

$\therefore \{u_n\}$ 发散

③ $p>1$ 且 $0<q\le 1$ 时. $\because \sum M_k$ 收敛, $\sum N_k$ 发散 $\therefore \sum u_n=\sum M_k-\sum N_k$ 发散, $\sum|u_n|=\sum M_k+\sum N_k$ 发散

$\therefore \{u_n\}$ 发散.

④ $0<p\le 1$ 且 $0<q\le 1$ 时.

1° 若 $p=q$, $u_n=(-1)^n\cdot\frac{1}{n^p}$. $\because \{\frac{1}{n^p}\}$ 单调趋于零, 由 Leibniz 判别法 $\sum u_n$ 收敛

同时 $|u_n|=\frac{1}{n^p}$ 是发散 $\therefore \{u_n\}$ 条件收敛.

2° 若 $p<q$:

$\therefore (2n-1)^p<(2n)^q$ $\therefore \frac{1}{(2n-1)^p}-\frac{1}{(2n)^q}>0$.

对于级数 $\sum\left(\frac{1}{(2n-1)^p}-\frac{1}{(2n)^q}\right)$, 为正项级数

$\because (2n)^p\left(\frac{1}{(2n-1)^p}-\frac{1}{(2n)^q}\right)=\left(\frac{2n}{2n-1}\right)^p-(2n)^{p-q}\to 1 \quad (n\to\infty)$

$\therefore \frac{1}{(2n-1)^p}-\frac{1}{(2n)^q}\sim\frac{1}{(2n)^p}$ $\therefore \sum\left(\frac{1}{(2n-1)^p}-\frac{1}{(2n)^q}\right)$ 发散 $\therefore \sum u_n$ 发散.

至于 $\sum|u_n|$, 由于其等于两个发散的正项级数之和. 故其发散.

综上: $\{u_n\}$ 发散.

3° 若 $p>q$: 同理 $\{u_n\}$ 发散.

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9. $\because$ $|a_k|\cdot|b_k| \le \frac{|a_k|^2+|b_k|^2}{2} = \frac{a_k^2+b_k^2}{2}$

$\therefore |a_1b_1|+|a_2b_2|+\cdots \le \frac{a_1^2+b_1^2}{2}+\frac{a_2^2+b_2^2}{2}+\cdots$

$\therefore \sum_{n=1}^{\infty}|a_nb_n| \le \frac{1}{2}\sum_{n=1}^{\infty}a_n^2+\frac{1}{2}\sum_{n=1}^{\infty}b_n^2$

$\because \sum_{n=1}^{\infty}a_n^2$、$\sum_{n=1}^{\infty}b_n^2$ 都收敛 $\therefore \frac{1}{2}(\sum a_n^2+\sum b_n^2)$ 收敛 $\therefore \sum_{n=1}^{\infty}|a_nb_n|$ 收敛

$\therefore \sum_{n=1}^{\infty}a_nb_n$ 绝对收敛.

10. $\because \{b_n\}$ 有界，设 $|b_n| \le M$.

$\therefore \sum_{n=1}^{\infty}|a_nb_n| \le M\cdot\sum_{n=1}^{\infty}|a_n|$

$\because \sum_{n=1}^{\infty}a_n$ 绝对收敛 $\therefore \sum_{n=1}^{\infty}|a_n|$ 收敛. $\therefore M\sum_{n=1}^{\infty}|a_n|$ 收敛

$\therefore \sum_{n=1}^{\infty}|a_nb_n|$ 收敛.

$\therefore \sum_{n=1}^{\infty}a_nb_n$ 绝对收敛.

11.

$\because \sum(a_{2n-1}+a_{2n})$ 收敛

$\therefore$ 对 $\forall \varepsilon>0$. $\exists N_1\in\mathbb{N}^*$, $n>N_1$ 时，对 $\forall p$ 有 $|a_{2n+1}+a_{2n+2}+a_{2n+3}+a_{2n+4}+\cdots+a_{2n+2p-1}+a_{2n+2p}|<\varepsilon$.

$\because \lim\limits_{n\to\infty}a_n=0$ $\exists N_2\in\mathbb{N}^*$, $n>N_2$ 时. 有 $|a_n|<\varepsilon$

取 $N=\max\{2N_1+2, N_2\}$ 对 $\forall n>N$ 及 $\forall p$.

① n、p 均为偶：$|a_{n+1}+a_{n+2}+\cdots+a_{n+p}|=|(a_{n+1}+a_{n+2})+\cdots+(a_{n+p-1}+a_{n+p})|<\varepsilon$

② n 为偶、p 为奇：$|a_{n+1}+a_{n+2}+\cdots+a_{n+p}|\leq|(a_{n+1}+a_{n+2})+\cdots+(a_{n+p-1}+a_{n+p})|+|a_{n+p+1}|<2\varepsilon$

③ n 为奇、p 为偶：$|a_{n+1}+a_{n+2}+\cdots+a_{n+p}|\leq|a_{n+1}|+|(a_{n+2}+a_{n+3})+\cdots+(a_{n+p-2}+a_{n+p-1})|+|a_{n+p}|<3\varepsilon$.

④ n、p 均为奇：$|a_{n+1}+a_{n+2}+\cdots+a_{n+p}|\leq|a_{n+1}|+|(a_{n+2}+a_{n+3})+\cdots+(a_{n+p-3})+(a_{n+p-2})|+|a_{n+p-1}|+|a_{n+p}|<4\varepsilon$.

综上，$\sum a_n$ 收敛

更换条件后："$\sum(a_{2n-1}+a_{2n})$ 绝对收敛且 $\lim\limits_{n\to\infty}a_n=0 \Rightarrow \sum a_n$ 绝对收敛" 错误

有以下反例：取 $a_n=(-1)^{n+1}\cdot\frac{1}{n}$.

显然，满足 $\lim\limits_{n\to\infty}a_n=0$，且 $\sum(\frac{1}{2n-1}-\frac{1}{n})$ 收敛

然而 $\sum|a_n|=\sum\frac{1}{n}$ 发散.

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12. 设$B_m=\sum_{k=1}^{m}B_k$. 由Abel求和公式，$\sum_{k=1}^{m}a_kb_k=a_mB_m+\sum_{k=1}^{m-1}(a_k-a_{k+1})B_k$.

由于$\sum b_n$收敛 $\therefore \lim_{m\to\infty}B_m$存在. 故$\{B_m\}$有界. 设$|B_m|\le M$，记$\lim_{n\to\infty}B_n=B$.

$\because \sum(a_n-a_{n-1})$绝对收敛 $\therefore \lim_{m\to\infty}\sum_{k=2}^{m}(a_k-a_{k-1})=\lim_{m\to\infty}(a_m-a_1)$存在. 记$\lim_{n\to\infty}a_n=A$

$\because |B_k(a_k-a_{k+1})|\le M|a_k-a_{k+1}|$，又$\sum(a_n-a_{n-1})$绝对收敛. $\therefore \sum|B_k(a_k-a_{k+1})|$绝对收敛

$\therefore \lim_{m\to\infty}\sum_{k=1}^{m-1}(a_k-a_{k-1})B_k$ 存在，设之为C

$\therefore \lim_{m\to\infty}\sum_{k=1}^{m}a_kb_k=AB+C$ 存在. 故级数$\sum a_kb_k$收敛.

13. $\because \lim_{n\to\infty}n\left(\frac{a_n}{a_{n+1}}-1\right)>0$. $\therefore \exists C>0$ s.t. $n>N$时，$n\left(\frac{a_n}{a_{n+1}}-1\right)>c$ 即$\frac{a_n}{a_{n+1}}>1+\frac{c}{n}$.

(即$n>N$后，$\forall \frac{a_n}{a_{n+1}}>1$. a_n递减) $\therefore$ 有$\ln a_n-\ln a_{n+1}>\ln(1+\frac{c}{n})=\frac{c}{n}-\frac{c^2}{2n^2}+o(\frac{1}{n^2})$

$$\left.\begin{aligned}\Rightarrow \ln a_{n+1}-\ln a_n &< -\frac{c}{n}+\frac{c^2}{2n^2}+o\left(\frac{1}{n^2}\right)<-\frac{c}{n}+\frac{c^2}{n^2}\\ \ln a_n-\ln a_{n-1} &< -\frac{c}{n-1}+\frac{c^2}{(n-1)^2}\\ &\vdots\\ \ln a_{N+1}-\ln a_N &< -\frac{c}{N}+\frac{c^2}{N^2}\end{aligned}\right\}\xrightarrow{\text{相加}}$$

$$\Rightarrow \ln a_{n+1}-\ln a_N<-c\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{N}\right)+c^2\left(\frac{1}{n^2}+\frac{1}{(n-1)^2}+\cdots+\frac{1}{N^2}\right)$$

由$\sum\frac{1}{n}$发散与$\sum\frac{1}{n^2}$收敛，右式$\to-\infty$ $(n\to\infty)$

$\therefore \ln a_{n+1}-\ln a_N\to-\infty$ $\therefore \ln a_{n+1}\to-\infty$ $\therefore \lim_{n\to\infty}a_n=0$

14. 由$\{a_n\}$递减且趋于零知$\{a_n\}$是正项级数

注意到 $\sum_{k=n+1}^{n+p} k(a_k-a_{k+1}) = (n+1)a_{n+1}-(n+1)a_{n+2}+(n+2)a_{n+2}-(n+2)a_{n+3}+(n+3)a_{n+3}-(n+3)a_{n+4}$

$+\cdots+(n+p)a_{n+p}-(n+p)a_{n+p+1}$

$=(n+1)a_{n+1}+a_{n+2}+a_{n+3}+a_{n+4}+\cdots+a_{n+p}-(n+p)a_{n+p+1}$

$=(n+1)a_{n+1}+\sum_{k=n+2}^{n+p}a_k-(n+p)a_{n+p+1}$.

由$\sum a_n$收敛. $\forall \varepsilon>0$. $\exists N\in N^*$, 对$\forall p\in N^*$, $a_{n+1}+\cdots+a_{n+p}<\varepsilon$ 由$\{a_n\}$递减, $pa_{n+p}<\varepsilon$.

$\therefore \lim_{n\to\infty} na_n=0$ (习题14-1)

$\therefore$对$\forall\varepsilon$. $\exists N_0\in N^*$, $n>N_0$时. $|na_n|<\varepsilon$.

又$\{a_n\}$收敛. $\therefore$对$\forall\varepsilon$, $\exists N_2\in N^*$, $n>N_2$时, $\forall p$. $|a_{n+2}+\cdots+a_{n+p}|<\varepsilon$.

取$N'=\max\{N_0,N_2\}$. $n>N'$时. 对$\forall\varepsilon$, $\forall p$. 只要$n>N'$,

$\sum_{k=n+1}^{n+p}|k(a_k-a_{k+1})| = \left|\sum_{k=n+1}^{n+p}k(a_k-a_{k+1})\right| = \left|(n+1)a_{n+1}+\sum_{k=n+2}^{n+p}a_k-(n+p)a_{n+p+1}\right| \le (n+1)a_{n+1}+\sum_{k=n+2}^{n+p}a_k+(n+p)a_{n+p+1}$

$<(n+1)a_{n+1}+\sum_{k=n+2}^{n+p}a_k+(n+p+1)a_{n+p+1}<\varepsilon+\varepsilon+\varepsilon=3\varepsilon$ $\therefore \sum n(a_n-a_{n+1})$收敛.

15.

由(14). $\sum_{k=n+2}^{n+p}a_k=\sum_{k=n+1}^{n+p}k(a_k-a_{k+1})-(n+1)a_{n+1}+(n+p)a_{n+p+1}$

$$=\underbrace{\sum_{k=n+1}^{n+p}k(a_k-a_{k+1})}_{\to 0\ (\text{由级数收敛})}-\underbrace{(n+1)\cdot a_{n+1}}_{\to 0\ (\text{由}na_n\text{收敛})}+\frac{n+p}{n+p+1}\cdot\underbrace{(n+p+1)a_{n+p+1}}_{\to 0\ (\text{由}na_n\text{收敛})}\to 0$$

由柯西收敛准则 $\sum a_n$收敛.

16. 1) $\sum a_n$ 收敛 $\to \sum a_n^2$ 收敛 [错误] 反例如 $a_n=(-1)^n\frac{1}{\sqrt{n}}$. $\sum a_n$ 收敛(由Lebniz) 而 $\sum a_n^2=\sum\frac{1}{n}$ 发散

2) $\sum a_n$ 收敛 $\to \sum a_n^3$ 收敛 [错误] 反例如: $\{a_n\}: 1,-1,\frac{1}{\sqrt[3]{2}},-\frac{1}{2\sqrt[3]{2}},-\frac{1}{2\sqrt[3]{2}},\frac{1}{\sqrt[3]{3}},-\frac{1}{3\sqrt[3]{3}},-\frac{1}{3\sqrt[3]{3}},-\frac{1}{3\sqrt[3]{3}},\frac{1}{\sqrt[3]{4}},\cdots$

$\sum a_n$ 收敛 (部分和中存在和为0周期) 而 $\sum a_n^3=1-1+\frac{1}{2}-2\cdot\frac{1}{2^4}+\frac{1}{3}-3\cdot\frac{1}{3^4}+\cdots=\sum\frac{1}{n}-\sum\frac{1}{n^3}$ 发散.

17. 1) $\sum a_n^2$ 收敛 $\to \sum a_n$ 收敛 [错误] 反例如 $a_n=\frac{1}{n^2}$.

2) $\sum a_n^2$ 收敛 $\to \sum a_n^3$ 收敛 [正确] 证明如下:

$\because \sum a_n^2$ 收敛 $\therefore \lim\limits_{n\to\infty} a_n^2=0$ $\therefore \lim\limits_{n\to\infty} a_n=0$ $\therefore \exists N, n>N$ 时 $|a_n|\le 1$

$\therefore |a_n^3|\le a_n^2$ 由于 $\sum a_n^2$ 收敛 $\therefore \sum|a_n^3|$ 收敛 $\therefore \sum a_n^3$ 绝对收敛 $\therefore \sum a_n^3$ 收敛.

3) $\sum a_n^2$ 收敛 $\to \sum a_n$ 与 $\sum(-1)^n a_n$ 中至少有一者收敛 [错误]

反例如 $a_n=\begin{cases}0 & n\text{为奇}\\ \frac{2}{n} & n\text{为偶}\end{cases}$. $\sum a_n^2$ 是 $\sum\frac{1}{n^2}$ 高阶无穷小收敛. 而无论是 $\{a_k\}$ 还是 $\{(-1)^k a_k\}$ 都与 $\sum\frac{1}{n}$ 同阶, 发散

18.

$$\left.\begin{array}{l} u_1,u_2,\cdots,u_{n-m},u_{n-m+1},u_{n-m+2},\cdots,u_n \\ u_1',u_2',\cdots,u_{n-m}',u_{n-m+1}',u_{n-m+2}',\cdots,u_n'\end{array}\right]\text{重排} f:\ u_k=u_{f(k)}'$$

由于重排后每一项离开原来不超过 m 个位置

$\therefore u_1,u_2,\cdots,u_{n-m}$ 重排后必是数列 $u_1',u_2',\cdots,u_n'$ 中的某一部分

$\therefore u_1',u_2',\cdots,u_n'$ 的组成: $\begin{cases}① u_1,u_2,\cdots,u_{n-m}\\ ② u_{n-m+1}, u_{n-m+2}, u_{n-m+3},\cdots,u_n,u_{n+1},\cdots,u_{n+m}\text{中的}m\text{项}.\\ \quad\text{记为}u_{k_1},u_{k_2},\cdots,u_{k_m}\end{cases}$

$$\therefore |(u_1+u_2+\cdots+u_n)-(u_1'+u_2'+\cdots+u_n')|=|(u_{n-m+1}+u_{n-m+2}+\cdots+u_n)-(u_{k_1}+u_{k_2}+\cdots+u_{k_m})|$$

$$<|u_{n-m+1}|+|u_{n-m+2}|+\cdots+|u_n|+|u_{k_1}|+|u_{k_2}|+\cdots+|u_{k_m}|$$

由于 $\sum u_n$ 收敛 $\therefore \lim\limits_{n\to\infty}u_n=0$. 上式 $<2m\varepsilon$.

$\therefore \lim\limits_{n\to\infty}\sum\limits_{k=1}^n u_n=\lim\limits_{n\to\infty}\sum\limits_{k=1}^n u_n'$ $\therefore$ 重排级数 $\sum u_n'$ 也收敛.

19. (1)

$\because \frac{n^3-1}{n^3+1} = \frac{n-1}{n+1} \cdot \frac{n^2+n+1}{n^2-n+1} = \frac{n-1}{n+1} \cdot \frac{(n+1)^2-(n+1)+1}{n^2-n+1}$

$\therefore \prod_{k=2}^{n} \frac{k^3-1}{k^3+1} = \prod_{k=2}^{n} \frac{k-1}{k+1} \cdot \prod_{k=2}^{n} \frac{(k+1)^2-(k+1)+1}{k^2-k+1} = \frac{2}{n(n+1)} \cdot \frac{n^2+n+1}{3} = \frac{2}{3}\left(1+\frac{1}{n^2+n}\right)$

$\therefore \prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \lim_{n\to\infty} \frac{2}{3}\left(1+\frac{1}{n^2+n}\right) = \frac{2}{3}$

(2)

$1+\frac{1}{n^2-1} = \frac{n^2}{n^2-1} = \frac{n}{n-1} \cdot \frac{n}{n+1}$

$\therefore \prod_{k=2}^{n}\left(1+\frac{1}{k^2-1}\right) = \prod_{k=2}^{n} \frac{k}{k-1} \cdot \frac{k}{k+1} = \left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1}\right) \cdot \left(\frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1}\right) = \frac{2n}{n+1}$

$\therefore \prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right) = \lim_{n\to\infty} \frac{2n}{n+1} = 2$

(3) $1+\frac{1}{2^n-2} = \frac{2^n-1}{2^n-2} = \frac{2^n-1}{2(2^{n-1}-1)} = \frac{1}{2} \cdot \frac{2^n-1}{2^{n-1}-1}$

$\therefore \prod_{k=2}^{n}\left(1+\frac{1}{2^k-2}\right) = \frac{1}{2^{n-1}} \cdot \left(\frac{2^2-1}{2-1} \cdot \frac{2^3-1}{2^2-1} \cdots \frac{2^n-1}{2^{n-1}-1}\right) = \frac{2^n-1}{2^{n-1}}$

$\therefore \prod_{n=2}^{\infty}\left(1+\frac{1}{2^n-2}\right) = \lim_{n\to\infty}\left(2-\frac{1}{2^{n-1}}\right) = 2$.

20. (1). $\because \{\frac{1}{\sqrt{n}}\}$ 递减趋于零. 由Leibniz判别法 $\sum a_n = \sum (-1)^{n+1} \cdot \frac{1}{\sqrt{n}}$ 收敛.

$\because a_n^2 = \frac{1}{n}$ 是发散级数, 由定理4, $\prod_{n=1}^{\infty}(1+a_n)$ 发散.

(2) 设 $P_n = \prod_{k=1}^{n}(1+a_n)$ n为偶时, $P_n = \prod_{k=1}^{2h}(1+a_n) = \prod_{k=1}^{h}\left(1+\frac{1}{\sqrt{k}}\right)\left(1+\frac{1}{k}-\frac{1}{\sqrt{k}}\right) = \prod_{k=1}^{h}\left(1+\frac{1}{k\sqrt{k}}\right)$

$\because \sum \frac{1}{k\sqrt{k}}$ 收敛, $\sum \frac{1}{k^3}$ 收敛 由定理4 $\prod_{k=1}^{h}\left(1+\frac{1}{k\sqrt{k}}\right) \to A \quad (h\to\infty)$

$\therefore P_n = \begin{cases} \prod_{k=1}^{h}\left(1+\frac{1}{k\sqrt{k}}\right) & n=2h \\ \left(1+\frac{1}{\sqrt{h+1}}\right)\prod_{k=1}^{h}\left(1+\frac{1}{k\sqrt{k}}\right) & n=2h+1 \end{cases}$ $\because \lim_{n\to\infty} P_n = \lim_{h\to\infty} P_h = A$ $\therefore \prod_{n=1}^{\infty}(1+a_n)$ 收敛.

同时. 由于 $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}} + \sum_{n=1}^{\infty}\frac{1}{2n}$, 其中 $\sum (-1)^{n+1}\frac{1}{\sqrt{n}}$ 收敛 (Leibniz) 而 $\sum \frac{1}{2n}$ 发散

$\therefore \sum_{n=1}^{\infty} a_n$ 发散.

21. 证明：$\because \frac{|x^{n+1}|}{(n+1)!}\cdot\frac{n!}{|x^n|}=\frac{|x|}{n+1}\to 0\ (n\to\infty)$ $\therefore \sum\frac{x^n}{n!}$ 绝对收敛 同理 $\sum\frac{y^n}{n!}$ 绝对收敛.

记 $C_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$

$$=\frac{1}{n!}\left(y^n+nxy^{n-1}+\frac{n!}{2!(n-2)!}\cdot x^2y^{n-2}+\cdots\cdots+x^n\right)=\frac{1}{n!}\left(y^n+C_n^1xy^{n-1}+C_n^2x^2y^{n-2}+\cdots+x^n\right)$$

$$=\frac{1}{n!}(x+y)^n$$

由柯西定理，$\sum C_n$ 收敛 且 $\sum\limits_{n=1}^{\infty}\frac{(x+y)^n}{n!}=\sum\limits_{n=1}^{\infty}\frac{x^n}{n!}\cdot\sum\limits_{n=1}^{\infty}\frac{y^n}{n!}$

22. 证明：记 $C_n=a_1a_n+a_2a_{n-1}+\cdots+a_na_1=\sum\limits_{k=1}^{n}(-1)^{n+1}\frac{1}{\sqrt{k}\cdot\sqrt{n+1-k}}$

由基本不等式 $\frac{a+b}{2}\geqslant\sqrt{ab}$，$\frac{1}{\sqrt{ab}}\geqslant\frac{2}{a+b}$ $\therefore \frac{1}{\sqrt{k(n+1-k)}}\geqslant\frac{2}{n+1}$

$$\sum_{k=1}^{n}\frac{1}{\sqrt{k(n+1)-k}}\geqslant\frac{2}{n+1}\cdot n=\frac{2n}{n+1}\geqslant 1$$

$\therefore \lim\limits_{n\to\infty}C_n=\lim\limits_{n\to\infty}(-1)^{n+1}\frac{1}{\sqrt{k}\cdot\sqrt{n+1-k}}\neq 0$ $\therefore C_n$ 发散.

23. 证：记 $C_n=a_1a_n+a_2a_{n-1}+\cdots+a_na_1$

$$C_n=\sum_{k=1}^{n}(-1)^k\cdot\frac{1}{k}\cdot(-1)^{n+1-k}\cdot\frac{1}{(n+1-k)}=(-1)^{n+1}\sum_{k=1}^{n}\frac{1}{k(n+1-k)}=\frac{(-1)^{n+1}}{n+1}\sum_{k=1}^{n}\left(\frac{1}{k}+\frac{1}{n+1-k}\right)$$

$$=2\cdot\frac{(-1)^{n+1}}{n+1}\sum_{k=1}^{n}\frac{1}{k}$$

记 $D_n=\frac{1}{n+1}\sum\limits_{k=1}^{n}\frac{1}{k}$. $\frac{D_{n+1}}{D_n}=\frac{n+1}{n+2}\cdot\frac{\sum\limits_{k=1}^{n+1}\frac{1}{k}}{\sum\limits_{k=1}^{n}\frac{1}{k}}=\frac{n+1}{n+2}+\frac{1}{n+2\sum\limits_{k=1}^{n}\frac{1}{k}}=\frac{n+1+\frac{1}{\sum_{k=1}^{n}\frac{1}{k}}}{n+2}<\frac{n+1+1}{n+2}=1$

$\therefore \{D_n\}$ 是递减数列

注意到 $\lim\limits_{n\to\infty}\frac{(1+\frac{1}{2}+\cdots+\frac{1}{n+1})-(1+\frac{1}{2}+\cdots+\frac{1}{n})}{(n+2)-(n+1)}=\lim\limits_{n\to\infty}\frac{\frac{1}{n+1}}{1}=0$

$\{n+1\}$ 递增且发散至 $+\infty$. 由 Stolz 定理 $\lim\limits_{n\to\infty}D_n=\lim\limits_{n\to\infty}\frac{1+\cdots+\frac{1}{n}}{n+1}=0$.

$\therefore \{D_n\}$ 是递减且趋于零的数列. 由 Leibniz 判别法 $\sum C_n=2\sum(-1)^{n+1}D_n$ 收敛

习题14-(B)

1. $x>1$ 求 $\sum_{n=1}^{\infty}\frac{n!}{(x+1)(x+2)\cdots\cdots(x+n)}$ 的和

解：令 $u_n=\frac{(n+1)!}{(x+1)(x+2)\cdots(x+n)}$

注意到 $u_{n-1}-u_n=\frac{n!}{(x+1)(x+2)\cdots(x+n-1)}-\frac{(n+1)!}{(x+1)(x+2)\cdots(x+n)}$

$=\frac{n!(x+n-n-1)}{(x+1)\cdots\cdots(x+n)}=(x-1)\cdot\frac{n!}{(x+1)\cdots(x+n)}=(x-1)a_n$

$\therefore \sum_{n=1}^{\infty}a_n=a_1+\frac{1}{x-1}\sum_{n=2}^{\infty}(u_{n-1}-u_n)$ 记 $S_n=\sum_{k=1}^{n}a_n$

$\therefore S_n=\frac{1}{x+1}+\frac{1}{x-1}(u_1-u_n)=\frac{1}{x+1}+\frac{2}{(x-1)(x+1)}-\frac{u_n}{x-1}$

对于 u_n：作 $\ln u_n=\sum_{k=1}^{n}\ln\frac{1+k}{x+k}=-\sum_{k=1}^{n}\ln\frac{k+x}{k+1}=-\sum_{k=1}^{n}\ln\left(1+\frac{x-1}{k+1}\right)$

当 $\ln(1+\frac{x-1}{k+1})=\frac{x-1}{k+1}+o(\frac{1}{k+1})$ $(k\to\infty)$

$\therefore \sum\ln(1+\frac{x-1}{k+1})\to+\infty$ $\therefore \ln u_n\to-\infty$

$\therefore u_n\to 0$

$\therefore S_n\longrightarrow\frac{1}{x+1}+\frac{2}{(x-1)(x+1)}=\frac{1}{x-1}$

2. 判断敛散：

(1) $\sum_{n=1}^{\infty}\frac{\sin(2\pi e n!)}{n^p}$ $(p>0)$

(1) $p>1$ 时．$\left|\frac{\sin(2\pi e n!)}{n^p}\right|\le\frac{1}{n^p}$ 绝对收敛

$0<p\le1$ 时．由 taylor 公式，$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{e^{\xi}}{(n+1)!}$ $(0<\xi<1)$

$\therefore \sin(2\pi e n!)$

$=\sin\left[2\pi n!\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{e^{\xi}}{(n+1)!}\right)\right]$

$=\sin\left[2\pi\left(n!+n!+\cdots+1+\frac{e^{\xi}}{n+1}\right)\right]$

$=\sin\frac{2\pi e^{\xi}}{n+1}$ $\therefore \sum_{n=1}^{\infty}\frac{\sin(2\pi e n!)}{n^p}=\sum_{n=1}^{\infty}\frac{\sin\frac{2\pi e^{\xi}}{n+1}}{n^p}$

由于 $n\to\infty$ 时 $\left|\frac{\sin\frac{2\pi e^{\xi}}{n+1}}{n^p}\right|\le\frac{2\pi e}{(n+1)n^p}\le\frac{2\pi e}{n^{p+1}}$ $\because p>0$ $\therefore p+1>1$

绝对收敛

(2) $\sum_{n=1}^{\infty} x^{1+\frac{1}{2}+\cdots+\frac{1}{n}}$ $(x>0)$

由于 $1+\frac{1}{2}+\cdots+\frac{1}{n}=\ln(1+n)+\gamma_n$ $(\lim_{n\to\infty}\gamma_n=r)$

$x\geq 1$时. 易得到 级数发散

$0<x<1$时. 令 $x=e^{-p}$ $(p>0)$

$\therefore x^{1+\cdots+\frac{1}{n}}=e^{-p[\ln(1+n)+\gamma_n]}=\frac{e^{-p\gamma_n}}{(1+n)^p}$

$\therefore \left|x^{1+\cdots+\frac{1}{n}}\right|\leq\frac{M}{(n+1)^p}$ $\begin{cases} p>1 & 收敛 \\ 0<p\leq 1 & 发散\end{cases}$ $\xRightarrow{p=-\ln x}$ $\begin{cases} x<\frac{1}{e}时 & 收敛 \\ x>\frac{1}{e}时 & 发散\end{cases}$

(3) $\sum_{n=1}^{\infty}\left(\sqrt[n]{a}-\sqrt{1+\frac{1}{n}}\right)$ $(a>0)$

令 $x=\frac{1}{n}$, $\sqrt[n]{a}-\sqrt{1+\frac{1}{n}}=a^x-(1+x)^{\frac{1}{2}}$

$=\left[1+x\ln a+\frac{\ln^2 a}{2}x^2+o(x^2)\right]-\left[1+\frac{x}{2}-\frac{x^2}{8}+o(x^2)\right]$

$=x(\ln a-\frac{1}{2})+x^2\left(\frac{\ln^2 a}{2}+\frac{1}{8}\right)+o(x^2)$

$=\frac{1}{n}(\ln a-\frac{1}{2})+\frac{1}{n^2}\left(\frac{\ln^2 a}{2}+\frac{1}{8}\right)+o(x^2)$

只有 $\ln a-\frac{1}{2}=0$ 时. 原级数收敛

即 $a=\sqrt{e}$时 级数收敛

$a\neq\sqrt{e}$时 发散

(4) $\sum_{n=2}^{\infty}\left(1-\frac{1}{n^p}\right)^n$ $(0<p<1)$

作 $\ln\left(1-\frac{1}{n^p}\right)^n=n\ln\left(1-\frac{1}{n^p}\right)$ 由于 $x\to 0$时 $\ln(1-x)<-x$

$\therefore n\ln\left(1-\frac{1}{n^p}\right)<-\frac{n}{n^p}=-n^{1-p}$

从而 $\left(1-\frac{1}{n^p}\right)^n<e^{-n^{1-p}}=\frac{1}{e^{n^{1-p}}}<\frac{C}{n^2}$ 由$\sum\frac{1}{n^2}$收敛

(由于 $e^{n^2}=1+n^2+\frac{n^4}{2}+o(n^4)>cn^2$) 级数收敛

(5) $\sum_{n=1}^{\infty}\left(1-\frac{\ln n}{n}\right)^{pn}$ $(p>0)$

$\because \ln(1-x)=-x-\frac{x^2}{2}+o(x^2)$

$\ln\left(1-\frac{\ln n}{n}\right)=-\frac{\ln n}{n}-\frac{\ln^2 n}{2n^2}+o\left(\frac{\ln^2 n}{n^2}\right)$

从而 $pn\ln\left(1-\frac{\ln n}{n}\right)=-\ln n^p-\frac{p\ln^2 n}{2n}+o\left(\frac{\ln^2 n}{n}\right)$

$\therefore \left(1-\frac{\ln n}{n}\right)^{pn}=\frac{1}{n^p}e^{o(1)}\sim\frac{1}{n^p}$

$\therefore p>1$时 收敛

$0<p\leq 1$时 发散

(6) $\sum_{n=1}^{\infty}\frac{(n+x)^n}{n^{n+x}}\quad(x>0)$

由于 $\frac{(n+x)^n}{n^{n+x}}\cdot n^x=\left(1+\frac{x}{n}\right)^n\to e^x$ $\quad\therefore \frac{(n+x)^n}{n^{n+x}}\sim\frac{e^x}{n^x}$ 其中，x为常数

$x>1$时：级数收敛

$0<x\le1$时：级数发散

(7) $\sum_{n=1}^{\infty}n^p(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1})\quad(p>0)$

$$n^p(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1})=n^p[(\sqrt{n+1}-\sqrt{n})-(\sqrt{n}-\sqrt{n-1})]=n^p\left[\frac{1}{\sqrt{n+1}+\sqrt{n}}-\frac{1}{\sqrt{n}+\sqrt{n-1}}\right]$$

$$=\frac{n^p}{(\sqrt{n+1}+\sqrt{n})(\sqrt{n}+\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}$$

$$\sim n^{p-\frac{3}{2}}\sim\frac{1}{n^{\frac{3}{2}-p}}$$

$\frac{3}{2}-p>1$ 即 $0<p<\frac{1}{2}$时：收敛

$\frac{3}{2}-p\le1$ 即 $p\ge\frac{1}{2}$时：发散

(8) $\sum_{n=0}^{\infty}\frac{a(a+d)(a+2d)\cdots(a+nd)}{b(b+d)(b+2d)\cdots(b+nd)}\quad(a>0,b>0,d>0)$

$a>b$ 发散

$a<b$ · 若$b=a+d$： $u_n=\frac{a}{a+(n+1)d}$ 发散

· $b\ne a+d$时：

$$u_n=\frac{1}{b-a-d}\left(\frac{a(a+d)\cdots(a+nd)}{b(b+d)\cdots(b+(n-1)d)}-\frac{a(a+d)\cdots(a+(n+1)d)}{b(b+d)\cdots(b+nd)}\right)\triangleq\frac{1}{b-a-d}(V_{n-1}-V_n)$$

$$\sum u_n\doteq S_n=\frac{a}{b}+\frac{1}{b-a-d}\left(\frac{a(a+d)}{b}-V_n\right)$$

其中 $V_n=a\cdot\frac{a+d}{b}\cdot\frac{a+d+d}{b+d}\cdots\frac{a+d+nd}{b+nd}$

考虑到 $\ln V_n=\ln a+\ln\frac{a+d}{b}+\cdots+\ln\frac{a+d+nd}{b+nd}$

由于 $\ln\frac{a+d+nd}{b+nd}=\ln\left(1+\frac{a+d-b}{b+nd}\right)$

· $a+d>b$： $\frac{a+d-b}{b+nd}\to+\infty$ $\quad\ln\frac{a+d+nd}{b+nd}\to+\infty\Rightarrow\therefore V_n\to+\infty$

$$S_n=\frac{a}{b}+\frac{1}{b-a-d}\left(\frac{a(a+d)}{b}-V_n\right)\to\infty\quad\text{发散}$$

· $a+d<b$： $\frac{a+d-b}{b+nd}\to-\infty$ $\quad\ln\frac{a+d+nd}{b+nd}\to-\infty\Rightarrow\therefore V_n\to0$

$$S_n=\frac{a}{b}+\frac{1}{b-a-d}\cdot\frac{a(a+d)}{b}\quad\text{收敛}$$

3. 设$f(x)$单调递减，$f(x)>0$。 $\lim_{x\to\infty}\frac{e^xf(e^x)}{f(x)}=\lambda$

求证：$\lambda<1$时 $\sum f(n)$收敛，$\lambda>1$时 $\sum f(n)$发散.

$\because f(x)\downarrow,\ f(x)>0$ 由积分判别法

$\therefore \sum f(n) \sim \int_1^n f(x)dx \sim \int_1^{a_n} f(x)dx$

构造$\{a_n\}$: $a_0=1,\ a_{n+1}=e^{a_n}$ ($a_n\uparrow +\infty$ 符合条件)

由于 $\int_1^{a_n} f(x)dx = \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k} f(x)dx$

令 $U_{n+1} = \int_{a_n}^{a_{n+1}} f(x)dx = \int_{e^{a_{n-1}}}^{e^{a_n}} f(x)dx \xlongequal{x=e^t} \int_{a_{n-1}}^{a_n} f(e^x)e^x dx$

① $\lambda<1$时

取 $\varepsilon>0$ s.t. $\lambda+\varepsilon<1$

$\lim\limits_{x\to\infty}\frac{e^x f(e^x)}{f(x)}=\lambda \Rightarrow \exists M>0$ s.t: $x>M$时, $\lambda-\varepsilon<\frac{e^x f(e^x)}{f(x)}<\lambda+\varepsilon<1$

$\lim\limits_{n\to\infty} a_n=\infty \Rightarrow \exists N\in\mathbb{N}$ s.t: $n>N$时, $a_{n-1}>M$

故此时在$[a_{n-1},a_n]$上有 $\frac{e^x f(e^x)}{f(x)}<\lambda+\varepsilon<1$ 即 $e^x f(e^x)<(\lambda+\varepsilon)f(x)$

$\therefore U_{n+1}=\int_{a_{n-1}}^{a_n} f(e^x)e^x dx < \int_{a_{n-1}}^{a_n}(\lambda+\varepsilon)f(x)dx = (\lambda+\varepsilon)U_n$

$\therefore \frac{U_{n+1}}{U_n}<\lambda+\varepsilon<1$ $\therefore \sum U_n$ 收敛 故 $\sum f(n)$ 收敛

② $\lambda>1$时

取 $\varepsilon>0$ s.t. $\lambda-\varepsilon>1$

$\lim\limits_{x\to\infty}\frac{e^x f(e^x)}{f(x)}=\lambda \Rightarrow \exists M>0$ s.t: $x>M$时, $\lambda+\varepsilon>\frac{e^x f(e^x)}{f(x)}>\lambda-\varepsilon>1$

$\lim\limits_{n\to\infty} a_n=\infty \Rightarrow \exists N\in\mathbb{N}$ s.t: $n>N$时, $a_{n-1}>M$

故此时在$[a_{n-1},a_n]$上有 $\frac{e^x f(e^x)}{f(x)}>\lambda-\varepsilon>1$ 即 $e^x f(e^x)>(\lambda-\varepsilon)f(x)$

$\therefore U_{n+1}=\int_{a_{n-1}}^{a_n} f(e^x)e^x dx > \int_{a_{n-1}}^{a_n}(\lambda-\varepsilon)f(x)dx = (\lambda-\varepsilon)U_n$

$\therefore \frac{U_{n+1}}{U_n}>\lambda-\varepsilon>1$ $\therefore \sum U_n$ 发散 故 $\sum f(n)$ 发散