第四章 导数

§4.1 导数概念

- 导数的定义.
- 可导的定义.
- 左、右导数的定义.

§4.2 导数计算

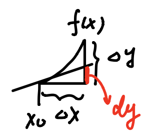
- 多项乘积求导：$(u_1u_2\cdots u_n)'=u_1'u_2\cdots u_n+u_1u_2'\cdots u_n+\cdots+u_1u_2\cdots u_n'$
- 反函数求导： $y=f(x)$ 在 (a,b) 连续且严格单调. 则：$\left.\begin{array}{l} y=f(x)\text{在}x_0\text{可导}. \\ f'(x_0)\neq 0. \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{反函数 } x=\varphi(y)\text{ 在点 }y_0=f(x_0)\text{ 可导} \\ \varphi'(y_0)=\dfrac{1}{f'(x_0)} \end{array}\right.$
- 三角函数求导：$(\arcsin x)'=\dfrac{1}{\sqrt{1-x^2}}$ $\quad(\arccos x)'=-\dfrac{1}{\sqrt{1-x^2}}$ $\quad(\arctan x)'=\dfrac{1}{1+x^2}$
- 隐函数求导：注意 y'.
- 参数方程求导：$\begin{cases} x=x(t) \\ y=y(t). \end{cases}$ 求 $\dfrac{dx}{dt}$、$\dfrac{dy}{dt}$. 则 $\dfrac{dy}{dx}=\dfrac{dy}{dt}\div\dfrac{dx}{dt}$

§4.3 高阶导数

- $\begin{cases} (\sin x)^{(n)}=\sin(x+\frac{n}{2}\pi) \\ (\cos x)^{(n)}=\cos(x+\frac{n}{2}\pi) \end{cases}$
- Leibniz公式：$(uv)^{(n)}=\sum\limits_{k=0}^{n}C_n^k\cdot u^{(k)}\cdot v^{(n-k)}$

§4.4 微分

- 一阶微分的形式不变性：$dy=f'(u)du$ $\qquad dy=y'\cdot dx.$
- 二阶微分不具形式不变性.



导数补充题

笔记P31. P33. P34. (4B-3)

练习4.1

1. (1) $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\frac{3+x+\Delta x}{3-x-\Delta x}-\frac{3+x}{3-x}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{6}{(3-x-\Delta x)(3-x)}=\frac{6}{(3-x)^2}$ $\therefore \frac{dy}{dx}\Big|_{x=2}=6$.

(2) $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\ln[1+(x+\Delta x)^2]-\ln(1+x^2)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\ln\frac{x^2+2x\Delta x+(\Delta x)^2+1}{x^2+1}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x(2x+\Delta x)}{x^2+1}\right)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}\frac{1}{\Delta x}\cdot\frac{\Delta x(2x+\Delta x)}{x^2+1}=\frac{2x}{x^2+1}$ $\therefore \frac{dy}{dx}\Big|_{x=1}=1$

(3) $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\sqrt{2(x+\Delta x)-1}-\sqrt{2x-1}}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{2\Delta x}{\Delta x(\sqrt{2(x+\Delta x)-1}+\sqrt{2x-1})}=\lim\limits_{\Delta x\to 0}\frac{2}{\sqrt{2(x+\Delta x)-1}+\sqrt{2x-1}}=\frac{1}{\sqrt{2x-1}}$

$\therefore \frac{dy}{dx}\Big|_{x=5}=\frac{1}{3}$

(4) $\frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{(x+\Delta x)^2\sin(x+\Delta x-2)-x^2\sin(x-2)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{2x^2\cos(x-2+\frac{\Delta x}{2})\cdot\sin\frac{\Delta x}{2}+\Delta x(2x+\Delta x)\sin(x+\Delta x)}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}x^2\cos(x-2+\frac{\Delta x}{2})+(2x+\Delta x)\sin(x+\Delta x-2)=x^2\cos(x-2)+2x\sin(x-2)$

$\therefore \frac{dy}{dx}\Big|_{x=2}=4$

2. $\because f(x)$在点x_0可导 $\therefore f(x)$在x_0左可导，右可导. 且$f'_-(x_0)=f'_+(x_0)$. 设$h>0$.

$\therefore \lim\limits_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}-\frac{f(x_0)-f(x_0-h)}{h}\right)=0$ $\therefore \lim\limits_{h\to 0}[f(x_0+h)+f(x_0-h)]=2f(x_0)$

$\therefore f'(x_0)=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim\limits_{h\to 0}\frac{f(x_0+h)-\frac{f(x_0+h)+f(x_0-h)}{2}}{h}=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$.

反例：对于$f(x)=|x|$. $x_0=0$. $\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$存在，但$f(x)$在点0不可导.

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3. $\left.\frac{dy}{dx}\right|_{x=0}=\frac{(\Delta x)^2\cdot D(\Delta x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\Delta x D(\Delta x)=\lim\limits_{t\to 0}t\cdot D(t)$.

由例6（课本P81）函数 $g(x)=xD(x)$ 在点0连续 $\therefore \lim\limits_{t\to 0}tD(t)=g(0)=0$.

$\therefore \left.\frac{dy}{dx}\right|_{x=0}=0$. $f(x)=x^2D(x)$ 在点0处可导.

4. $\because g(0)=0$. $\therefore \lim\limits_{\Delta x\to 0}\frac{g(\Delta x)-g(0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{g(\Delta x)}{\Delta x}=0$

$\therefore \left.\frac{dy}{dx}\right|_{x=0}=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x)-f(0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{f(\Delta x)}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{g(\Delta x)}{\Delta x}\cdot\sin\frac{1}{\Delta x}=0$ 即 $f'(0)=0$.

5. 由于 $\frac{d[f(g(x))]}{dx}=\frac{d[f(g(x))]}{dg(x)}\cdot\frac{dg(x)}{dx}=\lim\limits_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{g(x+\Delta x)-g(x)}\cdot\lim\limits_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$.

又 $\left.\frac{dg(x)}{dx}\right|_{x=0}=\lim\limits_{\Delta x\to 0}\frac{g(\Delta x)-g(0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\Delta x\cdot\sin\frac{1}{\Delta x}=\lim\limits_{x\to\infty}\frac{\sin x}{x}=0$

且由 $f(x)$ 在点0可导，$g(0)=0$ 知 $\left.\frac{d[f(g(x))]}{dg(x)}\right|_{x=0}$ 为有限值

$\therefore \left.\frac{d[f(g(x))]}{dx}\right|_{x=0}=0$.

1. $\because f(x)$ 在 x_0 可导. $\therefore f'_-(x_0)=f'_+(x_0)=f'(x_0)>0$. 取 $h>0$

由 $f'_-(x_0)=\lim\limits_{h\to 0^+}\frac{f(x_0)-f(x_0-h)}{h}>0$ 据极限保序性. $\exists h_1>0$, $0<h<h_1$ 时 $f(x_0)-f(x_0-h)>0$.

由 $f'_+(x_0)=\lim\limits_{h\to 0^+}\frac{f(x_0+h)-f(x_0)}{h}>0$ 据极限保序性. $\exists h_2>0$. $0<h<h_2$ 时 $f(x_0+h)-f(x_0)>0$.

记 $h=|x-x_0|$. 取 $\delta=\min\{h_1,h_2\}$. $\therefore$ $0<h<\delta$ 即 $x_0<x<x_0+\delta$ 时, $f(x)>x_0$.

$0<-h<\delta$ 即 $x_0-\delta<x<x_0$ 时. $f(x)<f(x_0)$.

7. ∵ $f(x)$ 在 x_0 可导 ∴ $f(x)$ 在 x_0 连续. ∵ $f(x_0)\neq 0$. 不妨设 $f(x_0)>0$.

∴ $\exists\,\delta>0$. 当 $|x-x_0|<\delta$ 时. $f(x)>0$. ∴ 当 $|x-x_0|<\delta$ 时, $f(x)=|f(x)|$.

∴ 由 $f(x)$ 在 x_0 可导 $\Leftrightarrow \lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ 存在 $\Leftrightarrow \lim\limits_{\Delta x\to 0}\dfrac{|f(x_0+\Delta x)|-|f(x_0)|}{\Delta x}$ 存在 $\Leftrightarrow |f(x)|$ 在 x_0 可导.

当 $f(x_0)=0$ 时. 结论可能不成立. 如 $|x|$ 在点 0 处不可导.

8. ∵ $f'(x_0)=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)}{\Delta x}$

令 $g(x)=|f(x)|$ ∴ $g'_+(x_0)=\lim\limits_{\Delta x\to 0^+}\left|\dfrac{f(x_0+\Delta x)}{\Delta x}\right|=f'(x_0)$, $g'_-(x_0)=-f'(x_0)$.

∵ $g(x)$ 在 x_0 可导 ∴ $g'_+(x_0)=g'_-(x_0)$ ∴ $f'(x_0)=0$.

9. 令 $\begin{cases}\log_a x_0=x_0\\ \dfrac{1}{x_0\ln a}=1\end{cases}\Rightarrow \dfrac{1}{\ln a}=\dfrac{-\ln(\ln a)}{\ln a}\Rightarrow -1=\ln(\ln a)\Rightarrow \begin{cases}a=e^{\frac{1}{e}}\\ x_0=e\end{cases}$ 切点为 (e,e)

10. $f'_+(0)=\lim\limits_{\Delta x\to 0^+}\dfrac{f(\Delta x)-f(0)}{\Delta x}=\lim\limits_{\Delta x\to 0^+}\dfrac{\Delta x|\Delta x|}{\Delta x}=\lim\limits_{\Delta x\to 0^+}\Delta x=0$

$f'_-(0)=\lim\limits_{\Delta x\to 0^-}\dfrac{f(\Delta x)-f(0)}{\Delta x}=\lim\limits_{\Delta x\to 0^-}\dfrac{\Delta x|\Delta x|}{\Delta x}=\lim\limits_{\Delta x\to 0^-}(-\Delta x)=0$

∵ $f'_+(0)=f'_-(0)=0$ ∴ $f(x)$ 在 0 可导.

练习4.2

1. 若 $f(x)=f(-x) \Rightarrow f'(x)=-f'(-x)$. $\therefore f'(x)$ 为奇函数

若 $f(x)=-f(-x) \Rightarrow f'(x)=f'(-x)$ $\therefore f'(x)$ 为偶函数

2. $(\sinh x)'=\frac{1}{2}(e^x-e^{-x})'=\frac{1}{2}(e^x+e^{-x})=\cosh x$

$(\cosh x)'=\frac{1}{2}(e^x+e^{-x})'=\frac{1}{2}(e^x-e^{-x})=\sinh x$

$(\tanh x)'=\frac{(\sinh x)'\cosh x-\sinh x\cdot(\cosh x)'}{\cosh^2 x}=\frac{\cosh^2 x-\sinh^2 x}{\cosh^2 x}=\frac{1}{\cosh^2 x}$

3. (1) $f'(x)=\frac{\frac{1}{x}\cdot\sqrt{x}-\ln x\cdot\frac{1}{2\sqrt{x}}}{x}=\frac{2-\ln x}{2x\sqrt{x}}$

(2) $f'(x)=3(x^2+1)^2e^{-2x}\sin 3x\cdot 2x+(x^2+1)^3\sin 3x\cdot e^{-2x}\cdot(-2)+(x^2+1)^3e^{-2x}\cdot 3\cos 3x$

$=(x^2+1)^2e^{-2x}\left[(1-2x^2)\sin 3x+(3x^2+3)\cos 3x\right]$

(3) $f'(x)=2^{\sin x}\ln 2\cdot\cos x-2^{\sin\frac{1}{x}}\ln 2\cdot\cos\frac{1}{x}\cdot\frac{1}{x^2}=2^{\sin x}\cos x\cdot\ln 2-\frac{1}{x^2}\cdot 2^{\sin\frac{1}{x}}\cos\frac{1}{x}\ln 2$

(4) $f'(x)=\frac{1+\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}=\frac{2\sqrt{x+\sqrt{x}}+1+\frac{1}{2\sqrt{x}}}{4\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}=\frac{4\sqrt{x}\cdot\sqrt{x+\sqrt{x}}+2\sqrt{x}+1}{8\sqrt{x}\cdot\sqrt{x+\sqrt{x}}\cdot\sqrt{x+\sqrt{x+\sqrt{x}}}}$

(5) $f'(x)=\frac{1}{2}\cdot\frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2}\cdot\frac{2(1-x^2)+2x\cdot 2x}{(1-x^2)^2}=\frac{1+x^2}{(1-x^2)^2+4x^2}=\frac{1+x^2}{(1+x^2)^2}=\frac{1}{1+x^2}$

(6) $f'(x)=\frac{d}{dx}\ln\left(1-\frac{2\sqrt{b-ac}}{\sqrt{ax+b}+\sqrt{b-ac}}\right)=\frac{\sqrt{ax+b}+\sqrt{b-ac}}{\sqrt{ax+b}-\sqrt{b-ac}}\cdot\frac{\sqrt{b-ac}}{(\sqrt{ax+b}+\sqrt{b-ac})^2}\cdot\frac{a}{\sqrt{ax+b}}$

$=\frac{(\sqrt{ax+b}+\sqrt{b-ac})^2}{a(x+c)}\cdot\frac{\sqrt{b-ac}}{(\sqrt{ax+b}+\sqrt{b-ac})^2}\cdot\frac{a}{\sqrt{ax+b}}=\frac{\sqrt{b-ac}}{(x+c)\sqrt{ax+b}}$

(7) $f'(x)=\begin{cases}2(x-a)(x-b)(2x-a-b) & x\in[a,b]\\ 0 & x\notin[a,b]\end{cases}$

(8) $|x|\le 1$ 时 $f'(x)=\frac{1}{1+x^2}$. $|x|>1$ 时 $f'(x)=\frac{1}{2}$. 由于 $\lim\limits_{x\to 1+}f(x)=\lim\limits_{x\to 1-}f(x)$. $\lim\limits_{x\to -1+}f(x)=\lim\limits_{x\to(-1)-}f(x)$.

$\therefore f(x)$ 在R上连续. 进一步 $f'(x)$ 在R上也连续 $\therefore f'(x)=\begin{cases}\frac{1}{1+x^2} & |x|\le 1\\ \frac{1}{2} & |x|>1\end{cases}$

4. (1) $(\sin x)^{\cos x} = e^{\cos x \ln(\sin x)}$.

$f'(x) = e^{\cos x \cdot \ln(\sin x)} \cdot \left[-\sin x \ln(\sin x) + \cos x \cdot \frac{\cos x}{\sin x}\right] = \sin x^{\cos x}\left[\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x)\right]$

(2) $y = a^{x^x} = a^{e^{x\ln x}}$ $\therefore f'(x) = a^{e^{x\ln x}} \cdot \ln a \cdot e^{x\ln x} \cdot (x + x \cdot \frac{1}{x}) = a^{x^x} \cdot x^x \cdot \ln a \cdot (1+x)$

(3) 注意: $x^{x^x} \neq (x^x)^x$. $x^{x^x} = x^{(x^x)}$ 而 $(x^x)^x = x^{x^2}$.

两边取对数. 有 $\ln y = x^x \cdot \ln x$ 当$x>1$时. $\ln(\ln y) = \ln(\ln x) + x\ln x$

$\therefore y' \cdot \frac{1}{y\ln y} = \frac{1}{x\ln x} + 1 + \ln x \Rightarrow y' = \left(\frac{1}{x\ln x} + 1 + \ln x\right) \cdot y\ln y = x^{x^x + x - 1}(1 + x\ln x + x\ln^2 x)$

经检验. y' 上式对 $0<x<1$ 也成立.

(4) $y = e^{2\ln x - \ln(1-x) + \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(1+x+x^2)}$

$f'(x) = \frac{x^2}{1-x}\sqrt{\frac{x+1}{1+x+x^2}} \cdot \left(\frac{2}{x} + \frac{1}{1-x} + \frac{1}{2(1+x)} - \frac{1+2x}{2(1+x+x^2)}\right)$

5. (1) $y' = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$

(2) $y' = e^x \cdot f'(x) \cdot e^{f(x)} + f(e^x) \cdot e^{f(x)} \cdot f'(x) = e^{f(x)} \cdot f'(x) \cdot (e^x + f(e^x))$

6. (1) $\frac{u(x)}{v(x)} = \tan y \Rightarrow \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{v^2(x)} = \frac{y'}{\cos^2 y} = (1 + \tan^2 y) y' = \left(\frac{u^2(x)}{v^2(x)} + 1\right) \cdot y' \Rightarrow y' = \frac{u'(x)v(x) - u(x)v'(x)}{u^2(x) + v^2(x)}$

(2) $y = v(x)^{\frac{1}{u(x)}} = e^{\frac{\ln v(x)}{u(x)}} \Rightarrow y' = \sqrt[u(x)]{v(x)} \cdot \frac{v'(x) \cdot \frac{u(x)}{v(x)} - u'(x) \cdot \ln v(x)}{u^2(x)} = \sqrt[u(x)]{v(x)} \cdot \frac{v'(x) \cdot u(x) - u'(x) v(x) \ln v(x)}{v(x)\, u^2(x)}$

(3) $y = \frac{\ln v(x)}{\ln u(x)} \Rightarrow y' = \frac{\frac{v'(x)}{v(x)} \cdot \ln u(x) - \frac{u'(x)}{u(x)} \ln v(x)}{\ln^2 u(x)} = \frac{v'(x)\, u(x) \ln u(x) - u'(x)\, v(x) \ln v(x)}{v(x)\, u(x) \ln^2 u(x)}$

(4) $y' = \sin[u(x)] \cdot e^{u(x)v(x)} \cdot (u'(x)v(x) + u(x)v'(x)) + e^{u(x)v(x)} \cdot \cos u(x) \cdot u'(x)$

$= e^{u(x)v(x)} \cdot \left[\sin u(x) \cdot \left(u'(x)v(x) + u(x)v'(x)\right) + \cos u(x) \cdot u'(x)\right]$

7. (1) $2y\cdot y'\cos x - y^2\sin x = 3a^2\cos 3x \Rightarrow y' = \dfrac{3a^2\cos 3x + y^2\sin x}{2y\cdot\cos x}$

(2) $y' - \cos x + \sin(x-y)\cdot(1-y') \Rightarrow y' = \dfrac{\cos x - \sin(x-y)}{1+\sin(x-y)}$

(3) $\dfrac{x\cdot y' - y}{x^2}\cdot\dfrac{1}{1+\left(\frac{y}{x}\right)^2} = \dfrac{1}{2}\cdot\dfrac{2x+2y\cdot y'}{x^2+y^2} \Rightarrow y' = \dfrac{x+y}{x-y}$

(4) $-2\sin 2x = e^{xy}(y+x\cdot y') + y'\ln x + \dfrac{y}{x} \Rightarrow y' = -\dfrac{y+xye^{xy}+2x\sin 2x}{x^2e^{xy}+x\ln x}$

8. (1) $\dfrac{dx}{dt} = 2e^{2t}\cos^2 t - e^{2t}\cdot 2\sin t\cos t \qquad \dfrac{dy}{dt} = 2e^{2t}\sin^2 t + e^{2t}\cdot 2\sin t\cos t$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{2\sin^2 t + 2\sin t\cos t}{2\cos^2 t - 2\sin t\cos t} = \dfrac{\sin^2 t + \sin t\cos t}{\cos^2 t - \sin t\cos t}$

(2) $\dfrac{dx}{dt} = a\left(\dfrac{\frac{1}{\cos^2\frac{t}{2}}\cdot\frac{1}{2}}{\tan\frac{t}{2}} - \sin t\right) = a\left(\dfrac{1}{\sin t} - \sin t\right)$

$\dfrac{dy}{dt} = a\cos t \qquad \therefore \dfrac{dy}{dx} = \dfrac{\cos t}{\frac{1}{\sin t} - \sin t} = \dfrac{\sin t\cos t}{1-\sin^2 t} = \tan t$

9. $\dfrac{dx}{dt} = -3a\cos^2 t\sin t \qquad \dfrac{dy}{dt} = 3a\sin^2 t\cos t \qquad \therefore \dfrac{dy}{dx} = -\tan t$

设点 $(a\cos^3 t, a\sin^3 t)$ 是曲线 (x,y) 上任一点.

对应的切线方程为 $y - a\sin^3 t = -\dfrac{\sin t}{\cos t}(x - a\cos^3 t)$

$$\left.\begin{array}{ll} \text{令 } x=0. & y = a\sin^3 t + a\sin t\cos^2 t = a\sin t \\ \text{令 } y=0 & x = a\sin^2 t\cos t + a\cos^3 t = a\cos t \end{array}\right\} \Rightarrow x^2+y^2=a^2$$

设所截长度为 l $\quad\therefore l = \sqrt{x^2+y^2} = |a| = a$ 是常数

练习4.3

1. (1) $y' = e^{-x^2}\cdot(-2x) + e^{-\frac{1}{x^2}}\cdot(2x^{-3})$ $\therefore y'' = 4x^2e^{-x^2} - 2e^{-x^2} + e^{-\frac{1}{x^2}}\cdot 4x^{-6} - 6x^{-4}e^{-\frac{1}{x^2}}$

即 $y'' = (4x^2-2)e^{-x^2} + (4x^{-6} - 6x^{-4})e^{-\frac{1}{x^2}}$

(2) $y' = 2x\arctan x + \frac{1+x^2}{1+x^2} = 2x\arctan x + 1$ $y'' = 2\arctan x + \frac{2x}{1+x^2}$

2. $f(x) = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases}$

$\left.\begin{array}{l} x>0\text{时 } f'(x) = 3x^2 \\ x<0\text{时 } f'(x) = -3x^2 \end{array}\right\}$ 由 $f'_+(0) = f'_-(0) = 0 \Rightarrow f'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0 \end{cases}$

$\left.\begin{array}{l} x>0\text{时 } f''(x) = 6x \\ x<0\text{时 } f''(x) = -6x \end{array}\right\}$ 由 $f''_+(0) = f''_-(0) = 0 \Rightarrow f''(x) = \begin{cases} 6x & x \geq 0 \\ -6x & x < 0 \end{cases}$

$\left.\begin{array}{l} x>0\text{时 } f'''(x) = 6 \\ x<0\text{时 } f'''(x) = -6 \end{array}\right\} \Rightarrow \begin{array}{l} f'''_+(0) = 6 \\ f'''_-(0) = -6 \end{array} \Rightarrow f'''_+(0) \neq f'''_-(0) \Rightarrow f'''(x)$在0处不存在.

3. (1) $y^{(98)} = \sum_{k=0}^{98} C_{98}^k \cdot (x^3)^{(k)} \cdot (\cos x)^{(98-k)}$ $\because k \geq 4$时 $(x^3)^{(k)} = 0$

$\therefore y^{(98)} = x^3\cdot\cos(x+49\pi) + 98\cdot 3x^2\cdot\cos(x+\frac{97}{2}\pi) + C_{98}^2\cdot 6x\cdot\cos(x+48\pi) + C_{98}^3\cdot 6\cdot\cos(x+\frac{95}{2}\pi)$

$= (6x\cdot C_{98}^2 - x^3)\cos x + (6C_{98}^3 - 294x^2)\sin x$

(2) $\because (x^9)^{(10)} = 0$

而 $\left(\frac{e^x}{x}\right)^{(10)} = \sum_{k=0}^{10} C_{10}^k\cdot(x^{-1})^{(k)}\cdot e^x = e^x\cdot\sum_{k=0}^{10}\frac{10!}{(10-k)!}\cdot\frac{(-1)^k}{x^{k+1}}$

$\therefore y^{(10)} = e^x\cdot\sum_{k=0}^{10}\left[\frac{10!}{(10-k)!}\cdot\frac{(-1)^k}{x^{k+1}}\right]$

4. (1) $y' = f'(x^2)\cdot 2x = 2xf'(x^2)$

$y'' = 2f'(x^2) + 4x^2 f''(x^2)$

$y''' = 4x\cdot f'''(x^2) + 8x f''(x^2) + 8x^3 f'''(x^2) = (4x+8x^3) f'''(x^2) + 8x f''(x^2)$

(2) $y' = -e^{-x} f'(e^{-x})$

$y'' = e^{-x} f'(e^{-x}) + e^{-2x} f''(e^{-x})$

$y''' = -e^{-x} f(e^{-x}) - e^{-2x} f''(e^{-x}) - 2e^{-2x} f''(e^{-x}) - e^{-3x} f'''(e^{-x}) = -e^{-x} f'(e^{-x}) - 3e^{-2x} f''(e^{-2x}) - e^{-3x} f'''(e^{-x})$

(3) $y' = \frac{f'(\ln x)}{x}$

$y'' = \frac{f''(\ln x) - f'(\ln x)}{x^2}$

$y''' = \frac{\frac{1}{x}\cdot x^2\cdot (f'''(\ln x) - f''(\ln x)) - 2x(f''(\ln x) - f'(\ln x))}{x^4} = \frac{f'''(\ln x) - 3f''(\ln x) + 2f'(\ln x)}{x^3}$

5. (1) $\frac{dx}{dt} = a(1-\cos t)$ $\quad \frac{dy}{dt} = a\sin t$ $\quad \therefore \frac{dy}{dx} = \frac{\sin t}{1-\cos t}$

$\therefore \frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dt} \div \frac{dx}{dt} = \frac{\cos t(1-\cos t) - \sin^2 t}{(1-\cos t)^2} \cdot \frac{1}{a(1-\cos t)} = -\frac{1}{a(1-\cos t)^2}$

(2) $\frac{dx}{dt} = f''(t)$ $\quad \frac{dy}{dt} = f'(t) + tf''(t) - f'(t) = tf''(t)$ $\quad \therefore \frac{dy}{dx} = t$

$\therefore \frac{d^2y}{dx^2} = \frac{d(\frac{dy}{dx})}{dt} \div \frac{dx}{dt} = \frac{1}{f''(t)}$

6. (1) $y+x\cdot y'-\frac{y'}{y}=0 \quad \therefore \frac{1-xy}{y}\cdot y'=y \Rightarrow (1-xy)y'=y^2 \Rightarrow \frac{dy}{dx}=y'=\frac{y^2}{1-xy}$

$$\frac{d^2y}{dx^2}=\frac{2y\cdot y'(1-xy)+y^2(y+xy')}{(1-xy)^2}=\frac{y^3+(2y-xy^2)y'}{(1-xy)^2}=\frac{3y^3-2xy^4}{(1-xy)^3}$$

(2) $3x^2+3y^2\cdot y'-3a(y+xy')=0 \quad \therefore (3y^2-3ax)y'=3ay-3x^2$

$$\therefore \frac{dy}{dx}=y'=\frac{ay-x^2}{y^2-ax}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{(ay'-2x)(y^2-ax)-(ay-x^2)(2y\cdot y'-a)}{(y^2-ax)^2}$$

$$=\frac{(2x^2y-a^2x-ay^2)\cdot y'+ax^2+a^2y-2xy^2}{(y^2-ax)^2}$$

$$=\frac{(2x^2y-a^2x-ay^2)(ay-x^2)+(y^2-ax)(ax^2+a^2y-2xy^2)}{(y^2-ax)^3}$$

$$=\frac{2ax^2y^2-2x^4y-a^3xy+a^2x^3-a^2x^3+ax^2y^2+ax^2y^2+a^2y^3-2xy^4-a^2x^3-a^3xy+2ax^2y^2}{(y^2-ax)^3}$$

$$=\frac{2a^3xy}{(ax-y^2)^3}$$

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练习4.4、

1. (1) $y'=\frac{dy}{dx}=\ln x+1$ $\quad\therefore dy=(1+\ln x)dx$ $\quad x=e^2$ 时. $dy=3dx$

(2) $y'=\frac{dy}{dx}=\frac{1}{2}\sqrt{x^2+a^2}+\frac{x^2}{2\sqrt{x^2+a^2}}+\frac{a^2}{2}\cdot\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)=\sqrt{x^2+a^2}$. $x=3a$ 时. $dy=\sqrt{10}|a|dx$.

2. (1) $y'=\frac{dy}{dx}=5^{\ln\tan x}\ln 5\cdot\frac{1}{\tan x}\cdot\frac{1}{\cos^2 x}$ $\quad\therefore dy=\frac{5^{\ln\tan x}\ln 5}{\sin x\cos x}dx$

(2) $y'=\frac{dy}{dx}=\frac{1}{2\sqrt{\arcsin x}}\cdot\frac{1}{\sqrt{1-x^2}}+2\arctan x\cdot\frac{1}{1+x^2}$ $\quad\therefore dy=\left(\frac{1}{2\sqrt{1-x^2}\sqrt{\arcsin x}}+\frac{2\arctan x}{1+x^2}\right)dx$

(3) $y'=\frac{dy}{dx}=\frac{2x\cos x^2(e^x-e^{-x})-\sin x^2\cdot(e^x+e^{-x})}{(e^x-e^{-x})^2}$ $\quad\therefore dy=\frac{(2x\cos x^2-\sin x^2)e^x-(2x\cos x^2+\sin x^2)e^{-x}}{(e^x-e^{-x})^2}dx$

(4) $y'=\frac{dy}{dx}=\frac{1}{(x-a)(x+a)}$ $\quad\therefore dy=\frac{1}{x^2-a^2}dx$

3. (1) $y'=u'vw+uv'w+uvw'+2u\cdot u'$ $\quad\therefore dy=vwdu+uwdv+uvdw+2u\cdot du$

(2) $y'=\frac{1}{\sqrt{u^2+v^2}}\cdot\frac{1}{2\sqrt{u^2+v^2}}\cdot(2u\cdot u'+2v\cdot v')=\frac{u\cdot u'+v\cdot v'}{u^2+v^2}$ $\quad\therefore dy=\frac{u\cdot du+v\cdot dv}{u^2+v^2}$

(3) $y'=e^{uv+w}\cdot(u'v+uv'+w')$ $\quad\therefore dy=e^{uv+w}(v\cdot du+u\cdot dv+dw)$

4. (1) $\frac{dy}{dx}=3e^{3u}\cdot\frac{du}{dx}$ $\quad\frac{du}{dx}=\frac{1}{2v}\cdot\frac{dv}{dx}$. $\quad\frac{dv}{dx}=3x^2-2$

$\therefore\frac{dy}{dx}=3e^{3u}\cdot\frac{1}{2v}\cdot(3x^2-2)=3e^{\frac{3}{2}\ln v}\cdot\frac{1}{2v}\cdot(3x^2-2)=\frac{3}{2}\sqrt{v}\cdot(3x^2-2)=\frac{3}{2}\sqrt{x^3-2x+5}(3x^2-2)$

$\therefore dy=\frac{3}{2}\sqrt{x^3-2x+5}(3x^2-2)dx$.

(2) $\frac{dy}{dx}=\frac{1}{\tan\frac{u}{2}}\cdot\frac{1}{\cos^2\frac{u}{2}}\cdot\frac{1}{2}\cdot\frac{du}{dx}$ $\quad\frac{du}{dx}=\frac{1}{\sqrt{1-v^2}}\cdot\frac{dv}{dx}$ $\quad\frac{dv}{dx}=-2\sin 2x$

$\therefore\frac{dy}{dx}=\frac{1}{2\sin\frac{u}{2}\cos\frac{u}{2}}\cdot\frac{1}{\sqrt{1-v^2}}\cdot(-2\sin 2x)=\frac{1}{\sin u}\cdot\frac{1}{\sqrt{1-v^2}}(-2\sin 2x)=\frac{-2\sin 2x}{v\cdot\sqrt{1-v^2}}$

$\therefore dy=\frac{-2\sin 2x}{\cos 2x\cdot|\sin 2x|}dx=\frac{-2}{\cos 2x}\cdot\operatorname{sgn}(\sin 2x)\,dx$

习题4.

1. 不妨设 $f'_+(a)>0$，$f'_-(b)>0$ 取 $\delta>0$. 则 $f(x)$ 在 $(a,a+\delta)$ 递增，在 $(b-\delta,b)$ 内也递增.

取 $x_1\in(a,a+\delta)$ $x_2\in(b-\delta,\delta)$ $\because f(x_1)>f(a)=0$，$f(x_2)<f(b)=0$

$\therefore f(x_1)\cdot f(x_2)<0$ 又$\because f(x)$ 在 $[a,b]$ 连续. 由根的存在定理. $\exists\xi\in(x_1,x_2)\subseteq(a,b)$，使 $f(\xi)=0$.

2. 由习题2A-14. $R(x)$ 在 $\mathbb{R}$ 上任一点的极限均为零. $\therefore R(x)$ 在所有无理点连续，有理点间断.

$\therefore$ 对 $\forall x_0\in\mathbb{Q}$，$R(x)$ 在 x_0 不可导.

对 $\forall x_0\in\mathbb{R}\setminus\mathbb{Q}$：不妨设 $x_0\in(0,1)$. $\because R(x_0)=0$. 只需证 $\lim\limits_{h\to0}\dfrac{R(x_0+h)}{h}$ 不存在.

① 取 $\{h_n\}\in\mathbb{Q}$ 且 $h_n\to0$. $x_0+h_n\notin\mathbb{Q}$ $\therefore \lim\limits_{n\to\infty}\dfrac{R(x_0+h_n)}{h_n}=0$

② 设 $x_0=0.\overline{x_1x_2\cdots x_n\cdots}$ $(\forall x_i\in\{0,1,2,\cdots,9\})$. 取 $h_n=0.\overline{x_1x_2\cdots x_n}-x_0$ 显然 $h_n\to0$

$\therefore R(x_0+h_n)=R\left(\dfrac{x_1\cdots x_n}{10^n}\right)\geq\dfrac{1}{10^n}$ 而 $h_n=0.\overline{\underbrace{0\cdots0}_{n个}x_{n+1}x_{n+2}\cdots}\leq\dfrac{1}{10^n}$

$\therefore \dfrac{R(x_0+h_n)}{h_n}\geq1$ 由保序性 $\lim\limits_{n\to\infty}\dfrac{R(x_0+h_n)}{h_n}\geq1$.

综上，$\exists$ 两个 $\to0$ 的数列 $\{h_n\}$ 使 $\lim\limits_{n\to\infty}\dfrac{R(x_0+h_n)}{h_n}$ 取到两个不同的值. 由海涅定理. $\lim\limits_{h\to0}\dfrac{R(x_0+h)}{h}$ 不存在

即 $R(x)$ 在 $[0,1]$ 不可导. 又$\because R(x)$ 是以周期为1的周期函数<注> $\therefore R(x)$ 在 $\mathbb{R}$ 上处处不可导.

<注> $R(x)$ 周期性证明：即证 $R(x+n)=R(x)$ 对 $\forall n\in\mathbb{Z}$ 与 $x\in\mathbb{R}$ 均成立

对 $\forall x\in\mathbb{R}\setminus\mathbb{Q}$ 显然成立 $R(x+n)=R(x)=0$

对 $\forall x\in\mathbb{Q}$，设 $x=\dfrac{p}{q}$. q,p 为互素整数 即 $(p,q)=1$ $\therefore\exists a,b\in\mathbb{Z}$ 使 $(p+nq,q)=ap+(an+b)q=(p,q)$

$\therefore p+nq$ 与 q 也互素 $\therefore R(x+n)=R\left(\dfrac{p+nq}{q}\right)=\dfrac{1}{q}=R\left(\dfrac{p}{q}\right)=R(x)$. 成立.

$\therefore R(x)$ 以1为周期.

<注2> $R^2(x)$ 在 $\mathbb{R}$ 上也是处处不可导的.

3. $\because \dfrac{f(b_n)-f(a_n)}{b_n-a_n}=\dfrac{f(b_n)-f(x_0)+f(x_0)-f(a_n)}{b_n-a_n}=\dfrac{f(b_n)-f(x_0)}{b_n-x_0}\cdot\dfrac{b_n-x_0}{b_n-a_n}+\dfrac{f(x_0)-f(a_n)}{x_0-a_n}\cdot\dfrac{x_0-a_n}{b_n-a_n}$.

由 $\lim\limits_{n\to\infty}a_n=x_0$，$a_n<x_0$ $\therefore \dfrac{f(x_0)-f(a_n)}{x_0-a_n}=f'_-(x_0)=f'(x_0)$

由 $\lim\limits_{n\to\infty}b_n=x_0$，$b_n>x_0$ $\therefore \dfrac{f(b_n)-f(x_0)}{b_n-x_0}=f'_+(x_0)=f'(x_0)$

$\therefore \lim\limits_{n\to\infty}\dfrac{f(b_n)-f(a_n)}{b_n-a_n}=f'(x_0)\cdot\lim\limits_{n\to\infty}\dfrac{b_n-x_0}{b_n-a_n}+f'(x_0)\cdot\lim\limits_{n\to\infty}\dfrac{x_0-a_n}{b_n-a_n}=f'(x_0)\cdot\lim\limits_{n\to\infty}\dfrac{b_n-x_0+x_0-a_n}{b_n-a_n}=f'(x_0)$ #

<注> 题目中"$a_n<x_0<b_n$"的条件是必要的，将之删去后结论不正确。

反例如 $f(x)=\begin{cases}x^2\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$ 取 $a_n=\dfrac{1}{2n\pi}$ $f(a_n)=0$ $b_n-a_n=-\dfrac{1}{4n(2n\pi+\frac{\pi}{2})}$

$b_n=\dfrac{1}{2n\pi+\frac{\pi}{2}}$ $f(b_n)=\dfrac{1}{(2n\pi+\frac{\pi}{2})^2}$ $f(b_n)-f(a_n)=\dfrac{1}{(2n\pi+\frac{\pi}{2})^2}$

$\therefore \lim\limits_{n\to\infty}\dfrac{f(b_n)-f(a_n)}{b_n-a_n}=-\dfrac{2}{\pi}$. 而 $f'(0)=0$. 两者不等.

4. $\because f'(0)=a$ $\therefore \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=a$. 即 $x\to 0$ 时，$f(x)\to ax$.

$\therefore n\to\infty$ 时，由于 $\dfrac{k}{n^2}\to 0$ $(k=1,2,\cdots,n)$ $\therefore f(\frac{k}{n^2})\to a\cdot\dfrac{k}{n^2}$

$\therefore x_n\to a\left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\cdots+\dfrac{n}{n^2}\right)=\dfrac{a}{n^2}\cdot\dfrac{n(n+1)}{2}=\dfrac{a}{2}\cdot\dfrac{n+1}{n}\to\dfrac{a}{2}$ 即 $\lim\limits_{n\to\infty}x_n=\dfrac{a}{2}$

5. $\lim\limits_{n\to\infty}(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})$ 由等价无穷小 $1+x\sim e^x$

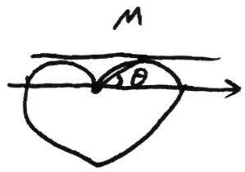
$=\lim\limits_{n\to\infty}e^{\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}}=\lim\limits_{n\to\infty}e^{\frac{1}{n^2}\cdot\frac{n(n+1)}{2}}=\lim\limits_{n\to\infty}e^{\frac{n+1}{2n}}=\sqrt{e}$

6. (1) $\because f'(a)=\lim\limits_{n\to\infty}\dfrac{f(a+\frac{1}{n})-f(a)}{\frac{1}{n}}$ $\therefore \lim\limits_{n\to\infty} f(a+\frac{1}{n})=\lim\limits_{n\to\infty}\dfrac{f'(a)}{n}+f(a)$

$\therefore \lim\limits_{n\to\infty}\left[\dfrac{f(a+\frac{1}{n})}{f(a)}\right]^n=\lim\limits_{n\to\infty}\left[\dfrac{\frac{f'(a)}{n}+f(a)}{f(a)}\right]^n=\lim\limits_{n\to\infty}\left[1+\dfrac{1}{n}\cdot\dfrac{f'(a)}{f(a)}\right]^{\frac{nf(a)}{f'(a)}\cdot\frac{f'(a)}{f(a)}}=e^{\frac{f'(a)}{f(a)}}$

(2) $\because f'(a)=\lim\limits_{t\to 0}\dfrac{f(a+\alpha t)-f(a+\beta t)}{(\alpha-\beta)t}$ $\therefore \lim\limits_{t\to 0}\dfrac{f(a+\alpha t)-f(a+\beta t)}{t}=(\alpha-\beta)f'(a)$

7. 由$r^2=ar+ar\cos\theta$ $\Rightarrow x^2+y^2=a\sqrt{x^2+y^2}+ax$ 两边求导得



$2x+2y\cdot y'=a+\dfrac{a(x+y\cdot y')}{\sqrt{x^2+y^2}}$ 点$M(\frac{3}{2}a,\frac{\pi}{3})$的直角坐标为$(\frac{3}{4}a,\frac{3}{4}\sqrt{3}a)$

代入有$\frac{3}{2}+\frac{3}{2}\sqrt{3}=1+\frac{2}{3}(\frac{3}{4}+\frac{3}{4}\sqrt{3}y')$ $\Rightarrow y'=0$ $\therefore$过M点的切线与x轴平行，与直线OM夹角为$\frac{\pi}{3}$.

8. $y'=\left(\dfrac{\sin x}{\cos x}\right)'=\dfrac{1}{\cos^2 x}=1+\tan^2 x=1+y^2$ $\therefore y''=2y\cdot y'=2y(1+y^2)=2y+2y^3$

$\therefore y'''=2y'+6y^2\cdot y'=(2+6y^2)(1+y^2)=2(1+y^2)(1+3y^2)$

9. (1) $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x}=\lim\limits_{x\to 0}\sin\frac{1}{x}\cdot|x|^{\alpha-1}$ 当$\alpha>1$时$f(x)$在0可导，进而在$\mathbb{R}$上可导.

(2) $f'(x)=\begin{cases}\alpha x^{\alpha-1}\sin\frac{1}{x}-x^{\alpha-2}\cos\frac{1}{x} & x>0\\ 0 & x=0\\ -\alpha x^{\alpha-1}\sin\frac{1}{x}+x^{\alpha-2}\cos\frac{1}{x} & x<0\end{cases}$ 当$\alpha>2$时$f(x)$在$\mathbb{R}$上连续可导.

(3) $f''(0)=\lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x}$ 当$f(x)$两次连续可导时，$f'(0)=0$. $\therefore \lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x}=\lim\limits_{x\to 0}\dfrac{f'(x)}{x}$

$\therefore f''_+(0)=\alpha x^{\alpha-2}\sin\frac{1}{x}-x^{\alpha-3}\cos\frac{1}{x}$. $f''_-(0)=-f''_+(0)$ $\therefore$当$\alpha>3$时$f(x)$在0上可导 即$f(x)$在$\mathbb{R}$上两次可导.

(4) $f''(x)=\begin{cases}\alpha(\alpha-1)x^{\alpha-2}\sin\frac{1}{x}-\alpha x^{\alpha-3}\cos\frac{1}{x}-(\alpha-2)x^{\alpha-3}\cos\frac{1}{x}-x^{\alpha-4}\sin\frac{1}{x} & x>0\\ 0 & x=0\\ -\alpha(\alpha-1)x^{\alpha-2}\sin\frac{1}{x}+\alpha x^{\alpha-3}\cos\frac{1}{x}+(\alpha-2)x^{\alpha-3}\cos\frac{1}{x}+x^{\alpha-4}\sin\frac{1}{x} & x<0\end{cases}$

$\therefore$当$\alpha>4$时$f(x)$在$\mathbb{R}$上两次连续可导.

10. ① 当 $n=1$ 时. $f'(0)=\lim\limits_{x\to 0}\dfrac{e^{-\frac{1}{x^2}}}{x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{x}}{e^{\frac{1}{x^2}}}\xlongequal{令y=\frac{1}{x}}\lim\limits_{y\to\infty}\dfrac{y}{e^{y^2}}=0$. $f'(0)=0$. 成立.

② 当 $n=k-1$ 时. 假设 $f^{(k-1)}(0)=0$

∵在 $x\neq 0$ 时. $f'(x)=P(\frac{1}{x})\cdot e^{-\frac{1}{x^2}}$. 其中 $P(\frac{1}{x})$ 为关于 $\frac{1}{x}$ 的某个多项式

∴ $f^{(k)}(0)=\lim\limits_{x\to 0}\dfrac{P(\frac{1}{x})\cdot e^{-\frac{1}{x^2}}}{x}\xlongequal{令y=\frac{1}{x}}\lim\limits_{y\to\infty}\dfrac{y\cdot P(y)}{e^{y^2}}=0$. $f^{(k)}(0)=0$ 成立.

根据①② 由数学归纳法知 ∀n, $f^{(n)}(0)=0$

11. (1) $n=1$ 时. $y'=\dfrac{\sqrt{1-x}+\frac{1+x}{2\sqrt{1-x}}}{1-x}=\dfrac{3-x}{2(1-x)\sqrt{1-x}}$

$n\geq 2$ 时. 令 $u=\dfrac{1}{\sqrt{1-x}}=(1-x)^{-\frac{1}{2}}$ ∴ $u^{(n)}=\dfrac{(2n-1)!!}{2^n}(1-x)^{-\frac{2n+1}{2}}$

由Leibniz公式 $y^{(n)}=C_n^0\cdot(1+x)\cdot u^{(n)}+C_n^1\cdot u^{(n-1)}=(1+x)\dfrac{(2n-1)!!}{2^n}(1-x)^{-\frac{2n+1}{2}}+n\cdot\dfrac{(2n-3)!!}{2^{n-1}}(1-x)^{-\frac{2n-1}{2}}$

(2) ∵ $\left(\dfrac{1}{x}\right)^{(n)}=\dfrac{(-1)^n\cdot n!}{x^{n+1}}$, $\left(\dfrac{1}{1-x}\right)^{(n)}=\dfrac{n!}{(1-x)^{n+1}}$ ∴ $y^{(n)}=\dfrac{(-1)^n\cdot n!}{x^{n+1}}+\dfrac{n!}{(1-x)^{n+1}}$

(3) $f'(x)=e^{ax}(a\sin bx+b\cos bx)=\sqrt{a^2+b^2}\cdot e^{ax}\sin(bx+\varphi)$. $(\varphi=\arctan\frac{b}{a})$

∴ $f''(x)=\sqrt{a^2+b^2}\cdot e^{ax}[a\sin(bx+\varphi)+b\cos(bx+\varphi)]=(a^2+b^2)e^{ax}\sin(bx+2\varphi)$

……

由归纳法可证得 $f^{(n)}(x)=(a^2+b^2)^{\frac{n}{2}}\cdot e^{ax}\sin(bx+n\varphi)$ $\varphi=\arctan\frac{b}{a}$

12. $\because y=\arcsin x$ $\therefore y'=\frac{1}{\sqrt{1-x^2}}$ $\therefore (y')^2\cdot(1-x^2)=1$.

两边求导得: $2y'\cdot y''(1-x^2)-2x\cdot(y')^2=0$ $\because y'\neq 0$ $\therefore y''(1-x^2)=x\cdot(y')^2$

两边各求n阶导. 由Leibniz公式可得 $y^{(n+2)}+ny^{(n+1)}\cdot(-2x)+C_n^2\cdot y^{(n)}\cdot(-2)=x\cdot y^{(n+1)}+n\cdot y^{(n)}$

取$x=0$. $\therefore f^{(n+2)}(0)=n^2\cdot f^{(n)}(0)$ $\because f'(0)=1$, $f''(0)=0$

$$\therefore f^{(n)}(0)=\begin{cases}1 & n=1\\ 0 & n\text{为偶}\\ [(n-2)!!]^2 & n\geq 3\text{且为奇}\end{cases}$$

13. $\varphi'(y)=\frac{1}{f'}\Rightarrow \varphi''(y)\cdot y'=-\frac{f''}{(f')^2}$ $\therefore \varphi''(y)=-\frac{f''}{(f')^3}$

$$\Rightarrow \varphi'''(y)\cdot f'=\frac{-f'''\cdot(f')^3+f''\cdot 3f''\cdot(f')^2}{(f')^6}=\frac{-f'''\cdot f'+3(f'')^2}{(f')^4}\Rightarrow \varphi'''(y)=\frac{3(f'')^2-f'\cdot f'''}{(f')^5}$$

14. $y=z\cdot x^{-\frac{1}{2}}$ $y'=z'x^{-\frac{1}{2}}-\frac{1}{2}zx^{-\frac{3}{2}}$ $y''=z''x^{-\frac{1}{2}}-z'x^{-\frac{3}{2}}+\frac{3}{4}zx^{-\frac{5}{2}}$

$\because y''+\frac{1}{x}y'+(1-\frac{1}{4x^2})y=0\Rightarrow z''x^{-\frac{1}{2}}-z'x^{-\frac{3}{2}}+\frac{3}{4}zx^{-\frac{5}{2}}+z'\cdot x^{-\frac{3}{2}}-\frac{z}{2}\cdot x^{-\frac{5}{2}}+(1-\frac{1}{4x^2})z\cdot x^{-\frac{1}{2}}=0$

$\Rightarrow \frac{z}{4}x^{-\frac{5}{2}}+z''x^{-\frac{1}{2}}+(1-\frac{1}{4x^2})z\cdot x^{-\frac{1}{2}}=0\Rightarrow z\cdot x^{-\frac{1}{2}}+z''x^{-\frac{1}{2}}=0\Rightarrow z''+z=0$

15. $\because y=(x+\sqrt{1+x^2})^m$ $\therefore \ln y=m\cdot\ln(x+\sqrt{1+x^2})$

$\therefore \frac{y'}{y}=m\cdot\frac{1}{x+\sqrt{1+x^2}}\cdot(1+\frac{x}{\sqrt{1+x^2}})=\frac{m}{\sqrt{1+x^2}}$ $\therefore (1+x^2)\cdot(y')^2=m^2y^2$

$\therefore 2y'\cdot y''(1+x^2)+2x\cdot(y')^2=2m^2y\cdot y'$. 显然$y'\neq 0$ $\therefore (1+x^2)y''+x\cdot y'=m^2y$