第七章 不定积分

<补充积分表>

(1) $\displaystyle\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C,\quad a \neq 0;$

(2) $\displaystyle\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C,\quad a \neq 0;$

(3) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C,\quad a > 0;$

(4) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}} = \ln|x+\sqrt{x^2\pm a^2}| + C,\quad a \neq 0;$

(5) $\displaystyle\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C,\quad a > 0;$

(6) $\displaystyle\int \sqrt{x^2\pm a^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln|x+\sqrt{x^2\pm a^2}| + C;$

(7) $\displaystyle\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C;$

(8) $\displaystyle\int \cot x\,\mathrm{d}x = \ln|\sin x| + C;$

(9) $\displaystyle\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C;$

(10) $\displaystyle\int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x| + C.$

<有理函数积分>

- (c) "$\dfrac{ax+b}{x^2+px+q}$", $p^2-4q<0$;

$$\frac{ax+b}{x^2+px+q} = \frac{a}{2}\cdot\frac{2x+p}{x^2+px+q} + \left(b-\frac{ap}{2}\right)\cdot\frac{1}{x^2+px+q}.$$

- (d) "$\dfrac{ax+b}{(x^2+px+q)^n}$", $p^2-4q<0, n=2,3,\cdots$.

$$\frac{ax+b}{(x^2+px+q)^n} = \frac{a}{2}\cdot\frac{2x+p}{(x^2+px+q)^n} + \left(b-\frac{ap}{2}\right)\cdot\frac{1}{(x^2+px+q)^n}.$$

对于(4)式右端第二项, 令$t = x+\dfrac{p}{2}$, $d = \dfrac{1}{2}\sqrt{4q-p^2}$, 类似于(3)式可得

$$\int\frac{\mathrm{d}x}{(x^2+px+q)^n} = \int\frac{\mathrm{d}t}{(t^2+d^2)^n}.\qquad I_n = \int\frac{\mathrm{d}x}{(x^2+a^2)^n},\ n\in\mathbb{N}^*,\ a\neq 0.$$

解 当$n>1$时, 改写后分部积分, 得到

$$\begin{aligned}
I_n &= \frac{1}{a^2}\int\frac{x^2+a^2-x^2}{(x^2+a^2)^n}\mathrm{d}x = \frac{1}{a^2}I_{n-1} - \frac{1}{a^2}\int\frac{x^2}{(x^2+a^2)^n}\mathrm{d}x\\
&= \frac{1}{a^2}I_{n-1} + \frac{1}{2(n-1)a^2}\int x\left[\frac{1}{(x^2+a^2)^{n-1}}\right]'\mathrm{d}x\\
&= \frac{1}{a^2}I_{n-1} + \frac{1}{2(n-1)a^2}\left[\frac{x}{(x^2+a^2)^{n-1}} - \int\frac{\mathrm{d}x}{(x^2+a^2)^{n-1}}\right]\\
&= \frac{2n-3}{2(n-1)a^2}I_{n-1} + \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}}.
\end{aligned}$$

易见, 这是I_n的一个递推公式.

7h $\displaystyle\frac{f(x)}{g(x)} = \sum_{i=1}^{s}\left[\frac{a_{i1}}{x-\alpha_i} + \frac{a_{i2}}{(x-\alpha_i)^2} + \cdots + \frac{a_{in_i}}{(x-\alpha_i)^{n_i}}\right] + \sum_{j=1}^{t}\left[\frac{b_{j1}x+c_{j1}}{x^2+p_jx+q_j} + \frac{b_{j2}x+c_{j2}}{(x^2+p_jx+q_j)^2} + \cdots + \frac{b_{jm_j}x+c_{jm_j}}{(x^2+p_jx+q_j)^{m_j}}\right]$

练习7.1

1.(1) $y=\begin{cases}\frac{x^3}{3}-\frac{2}{3}, & x\le -1\\ x & |x|<1\\ \frac{x^3}{3}+\frac{2}{3} & x\ge 1\end{cases}+C$

(2). $e^{-|x|}=\begin{cases}e^{-x} & x\ge 0\\ e^{x} & x<0\end{cases}$ $\quad y=\begin{cases}e^{x} & x<0\\ -e^{-x}+2 & x\ge 0\end{cases}+C$ (C为任意常数)

2.(1) $\because f'(x)=\frac{1}{4}x$ $\therefore f(x)=\frac{x^2}{8}+C$

$\because f(2)=\frac{5}{2}$ $\therefore \frac{1}{2}+C=\frac{5}{2}$ $\therefore C=2$ $\quad \therefore f(x)=\frac{x^2}{8}+2.$

(2) $f'(x)=\begin{cases}1 & x\le 0\\ e^{x} & x>0.\end{cases}$

$\because e^0+C_2=C_1=0$

$\therefore C_1=0, C_2=-1$

$\therefore f(x)=\begin{cases}x+C_1 & x\le 0\\ e^{x}+C_2 & x>0\end{cases}$ $\quad \therefore f(x)=\begin{cases}x & x\le 0\\ e^{x}-1 & x>0\end{cases}$

南開大學 作业纸

系别________ 班级________ 姓名________ 第 页

2.

(1)
$$\int x^2(1-x)^3dx = \int x^2(1-3x+3x^2-x^3)dx = \int(-x^5+3x^4-3x^3+x^2)dx = -\frac{x^6}{6}+\frac{3x^5}{5}-\frac{3x^4}{4}+\frac{x^3}{3}+C.$$

(2)
$$\int\left(\frac{1-x}{x}\right)^2dx = \int\frac{x^2-2x+1}{x^2}dx = \int\left(1-\frac{2}{x}+\frac{1}{x^2}\right)dx = \int(1-2x^{-1}+x^{-2})dx = x-\frac{1}{x}-2\ln x+C$$

(3)
$$\int\frac{x^2+1}{x^2-1}dx = \int\left(\frac{x^2-1+2}{x^2-1}\right)dx = \int\left(1+\frac{2}{x^2-1}\right)dx = \int\left(1+\frac{1}{x-1}-\frac{1}{x+1}\right)dx = x+\ln|x-1|-\ln|x+1|+C$$

(4)
$$\int\frac{x+1}{\sqrt{x}}dx = \int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int(x^{\frac{1}{2}}+x^{-\frac{1}{2}})dx = \frac{2}{3}x\sqrt{x}+2\sqrt{x}+C$$

(5)
$$\int\left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx = \int x^{\frac{3}{4}}(1-x^{-2})dx = \int(x^{\frac{3}{4}}-x^{-\frac{5}{4}})dx = \frac{4}{7}x\cdot\sqrt[4]{x^3}+\frac{4}{\sqrt[4]{x}}+C.$$

(6).
$$\int\left(\sqrt{\frac{1+x}{1-x}}+\sqrt{\frac{1-x}{1+x}}\right)dx = \int\frac{2dx}{\sqrt{1-x^2}} = 2\arcsin x+C$$

(7)
$$\int\frac{2^{x+1}-5^{x-1}}{10^x}dx = \int\left(\frac{2}{5^x}-\frac{1}{5\cdot 2^x}\right)dx = \int\left[2\cdot\left(\frac{1}{5}\right)^x-\frac{1}{5}\left(\frac{1}{2}\right)^x\right]dx = -\frac{2}{\ln 5}\cdot\frac{1}{5^x}+\frac{1}{5\cdot\ln 2}\cdot\frac{1}{2^x}+C$$

(8)
$$\int\cot^2x\,dx = \int\frac{\cos^2x}{\sin^2x}dx = \int\frac{1-\sin^2x}{\sin^2x}dx = \int(\csc^2x-1)dx = -\cot x-x+C$$

(1).
$$\int (2x-3)^{10}dx = \frac{1}{2}\int 2\cdot(2x-3)^{10}dx = \frac{1}{2}\cdot\frac{(2x-3)^{11}}{11} = \frac{(2x-3)^{11}}{22}+C$$

(2)
$$\int\frac{dx}{\sqrt{2-5x}} = -\frac{1}{5}\int\frac{-5dx}{\sqrt{2-5x}} = -\frac{1}{5}\cdot 2\sqrt{2-5x} = -\frac{2}{5}\sqrt{2-5x}+C$$

(3)
$$\int\frac{dx}{2+3x^2} = \frac{1}{2}\int\frac{dx}{1+(\frac{\sqrt6}{2}x)^2} = \frac{1}{2}\cdot\frac{2}{\sqrt6}\cdot\int\frac{d\frac{\sqrt6}{2}x}{1+(\frac{\sqrt6}{2}x)^2} = \frac{\sqrt6}{6}\arctan\frac{\sqrt6}{2}x+C$$

(4)
$$\int\frac{dx}{2-3x^2} = \frac{1}{2}\int\frac{dx}{(1+\frac{\sqrt6}{2}x)(1-\frac{\sqrt6}{2}x)} = \frac{1}{4}\int\left(\frac{1}{1-\frac{\sqrt6}{2}x}+\frac{1}{1+\frac{\sqrt6}{2}x}\right)dx$$
$$= \frac{1}{4}\cdot\frac{2}{\sqrt6}\cdot\int\left(\frac{\frac{\sqrt6}{2}}{\frac{\sqrt6}{2}x+1}-\frac{\frac{\sqrt6}{2}}{\frac{\sqrt6}{2}x-1}\right)dx = \frac{\sqrt6}{12}\ln\left|\frac{\frac{\sqrt6}{2}x+1}{\frac{\sqrt6}{2}x-1}\right|+C$$

(5)
$$\int\csc x\,dx = \int\frac{\sin x dx}{\sin^2 x} = -\int\frac{d\cos x}{1-\cos^2 x}$$
$$= -\frac{1}{2}\int\left(\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\right)d\cos x = -\frac{1}{2}(\ln|\cos x+1|-\ln|\cos x-1|)+C$$
$$= \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|+C = \ln|\csc x-\cot x|+C$$

(6)
$$\int(x+2)\sin(x^2+4x+5)dx = \frac{1}{2}\int 2(x+2)\sin[(x+2)^2+1]dx$$
$$= \frac{1}{2}\int\sin[(x+2)^2+1]\,d(x+2)^2 = -\frac{1}{2}\cos(x^2+4x+5)+C$$

(7).
$$\int\frac{1}{x^2}\sin\frac{1}{x}dx = -\int -\frac{1}{x^2}\sin\frac{1}{x}dx = -\int\sin\frac{1}{x}\,d\frac{1}{x} = \cos\frac{1}{x}+C$$

(8) $\int \frac{dx}{x\sqrt{x^2+1}} = \frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{x^2+1}} = \frac{1}{2}\int \frac{dx^2}{x^2\sqrt{x^2+1}}$ （令$a=\sqrt{x^2+1}$）

$= \frac{1}{2}\int \frac{d(a^2-1)}{a^3-a} = \frac{1}{2}\int \frac{2a\,da}{a^3-a} = \int \frac{da}{a^2-1} = \frac{1}{2}\int (\frac{1}{a-1}-\frac{1}{a+1})da = \frac{1}{2}\ln\left|\frac{a-1}{a+1}\right|+C$

$= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|+C = \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right|+C$

(9) 令$x=\tan^2 a$ $\therefore a=\arctan\sqrt{|x|}$

$\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{d\tan^2 a}{\tan a \cdot \sec^2 a} = \int \frac{2\tan a \cdot \sec^2 a\,da}{\tan a \sec^2 a} = \int 2da = 2a+C = 2\arctan\sqrt{|x|}+C$

(10) 令$\sqrt{3-x}=\sqrt{5}\sin a$. 即$\sqrt{x+2}=\sqrt{5}\cos a$ $\therefore x=5\cos^2 a-2,\ a=\arccos\sqrt{\frac{x+2}{5}}$

$\int \frac{dx}{\sqrt{(x+2)(3-x)}} = \int \frac{d(5\cos^2 a-1)}{5\sin a\cos a} = \int \frac{-10\sin a\cos a\,da}{5\sin a\cos a} = \int -2da = -2a+C$

$= -2\arccos\sqrt{\frac{x+2}{5}}+C$

(11) $\int xe^{-x^2}dx = -\frac{1}{2}\int (-2xe^{-x^2})dx = -\frac{1}{2}\int e^{-x^2}d(-x^2) = -\frac{1}{2}e^{-x^2}+C$

(12) $\int \frac{e^{3x}+1}{e^x+1}dx = \int (e^{2x}-e^x+1)dx = \frac{1}{2}e^{2x}-e^x+x+C$

(13) $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x dx}{e^{2x}+1} = \int \frac{de^x}{e^{2x}+1} = \arctan e^x+C$

(14) $\int \frac{dx}{\sqrt{1+e^{2x}}} = \int \frac{de^x}{e^x\sqrt{1+e^{2x}}} = \ln \frac{\sqrt{e^{2x}+1}-1}{e^x} + C$ （由(8)可知）

(15) $\int \frac{1}{x}\ln^2 x dx = \int \ln^2 x \, d\ln x = \frac{1}{3}\ln^3 x + C$

(16) $\int \frac{dx}{x\ln x \ln(\ln x)} = \int \frac{1}{\ln(\ln x)} d\ln(\ln x) = \ln|\ln(\ln x)| + C$

(17) $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}dx = \int \frac{d(\sin x-\cos x)}{\sqrt[3]{\sin x-\cos x}} = \frac{3}{2}\cdot(\sin x-\cos x)^{\frac{2}{3}}+C = \frac{3}{2}\sqrt[3]{1-\sin 2x} + C$

(18) $\int \frac{\arctan x}{1+x^2}dx = \int \arctan x \, d\arctan x = \frac{\arctan^2 x}{2} + C$

(19) $\int \frac{\sin x\cos x}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}dx = \int \frac{-\frac{1}{4}\cdot(-2\sin 2x)dx}{\sqrt{\frac{b^2+a^2}{2}+\frac{b^2-a^2}{2}\cos 2x}} = -\frac{1}{4}\int \frac{d\cos 2x}{\sqrt{\frac{b^2+a^2}{2}+\frac{b^2-a^2}{2}\cos 2x}}$

$= \frac{1}{2(a^2-b^2)}\int \frac{d\frac{b^2-a^2}{2}\cos 2x}{\sqrt{\frac{b^2+a^2}{2}+\frac{b^2-a^2}{2}\cos 2x}} = \frac{1}{a^2-b^2}\sqrt{\frac{b^2+a^2}{2}+\frac{b^2-a^2}{2}\cos 2x} + C$

$= \frac{1}{a^2-b^2}\sqrt{a^2\sin^2 x+b^2\cos^2 x} + C$

(20) $\int \sqrt{\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}}dx = \int \frac{\sqrt{\ln(x+\sqrt{1+x^2})}}{\sqrt{1+x^2}}dx =$

$= \int \sqrt{\ln(x+\sqrt{1+x^2})}\cdot\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}dx = \int \sqrt{\ln(x+\sqrt{1+x^2})}\cdot\left[\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)\right]dx$

$= \int \sqrt{\ln(x+\sqrt{1+x^2})}\cdot d\ln(x+\sqrt{1+x^2}) = \frac{2}{3}\ln^{\frac{3}{2}}(x+\sqrt{1+x^2}) + C$

(21)

$$\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}dx = \frac{1}{2}\int\frac{2}{(1-x)(1+x)}\ln\frac{1+x}{1-x}dx = \frac{1}{2}\int\frac{1-x+1+x}{(1-x)(1+x)}\ln\frac{1+x}{1-x}dx$$

$$=\frac{1}{2}\int\ln\frac{1+x}{1-x}d\left(\ln\frac{1+x}{1-x}\right) = \frac{1}{4}\cdot\ln^2\frac{1+x}{1-x}+C$$

(22)

$$\int\frac{x^{14}dx}{(x^5+1)^4} = \frac{1}{5}\int\frac{(x^5)^2dx^5}{(x^5+1)^4} = -\frac{1}{5}\cdot\int\frac{\frac{-1}{(x^5)^2}dx^5}{(1+\frac{1}{x^5})^4} = -\frac{1}{5}\int\frac{d(1+\frac{1}{x^5})}{(1+\frac{1}{x^5})^4}$$

$$=\frac{1}{15}(1+\frac{1}{x^5})^{-3}+C = \frac{x^{15}}{15(x^5+1)^3}+C$$

(23) $$\int\frac{dx}{x(x^n+1)} = \frac{1}{n}\int\frac{nx^{n-1}dx}{x^n(x^n+1)} = \frac{1}{n}\int\frac{dx^n}{x^n(x^n+1)} = \frac{1}{n}\int(\frac{1}{x^n}-\frac{1}{x^n+1})dx^n = \frac{1}{n}\ln\left|\frac{x^n}{x^n+1}\right|+C$$

(24) $$\int\frac{\cos x}{\sqrt{2+\cos^2x}}dx = \int\frac{d\sin x}{\sqrt{3-\sin^2x}} = \frac{\sqrt3}{3}\int\frac{d\sin x}{\sqrt{1-(\frac{\sin x}{\sqrt3})^2}} = \int\frac{d\frac{\sin x}{\sqrt3}}{\sqrt{1-(\frac{\sin x}{\sqrt3})^2}} = \arcsin(\frac{\sqrt3}{3}\sin x)+C$$

(25) 令 $\sqrt{2-5x}=t$ $\therefore x=\frac{2-t^2}{5}$

$$\therefore\int x\sqrt{2-5x}dx = \int\frac{t(2-t^2)}{5}d\frac{2-t^2}{5} = \int-\frac{2t^2}{25}(2-t^2)dt = \frac{2}{25}\int(t^4-2t^2)dt$$

$$=\frac{2}{125}t^5-\frac{4}{75}t^3+C = \frac{2}{375}(15x^2-10x-8)\sqrt{2-5x}+C$$

(26). 令 $\sqrt[3]{1+x^2}=t$ $\therefore x^2=t^3-1$ $\because t\geq1$ $\therefore x=\sqrt{t^3-1}$

$$\therefore\int x^3\sqrt[3]{1+x^2}dx = \int t(t^3-1)^{\frac{3}{2}}d\sqrt{t^3-1} = \int t\cdot(t^3-1)^{\frac{3}{2}}\cdot\frac{1}{2}\cdot(t^3-1)^{-\frac{1}{2}}\cdot3t^2dt$$

$$=\int\frac{3}{2}t^3(t^3-1)dt = \frac{3}{2}\int(t^6-t^3)dt = \frac{3t^7}{14}-\frac{3t^4}{8}+C = \frac{3}{56}(4x^2-3)\cdot\sqrt[3]{(1+x^2)^4}+C$$

(27). $\displaystyle\int\frac{x\,dx}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}=\int\frac{x\,dx}{\sqrt{(1+x^2)(1+\sqrt{1+x^2})}}=\int\frac{x}{\sqrt{1+x^2}}\cdot\frac{dx}{\sqrt{1+\sqrt{1+x^2}}}$

$\displaystyle=\int\frac{d\sqrt{1+x^2}}{\sqrt{1+\sqrt{1+x^2}}}=2\sqrt{1+\sqrt{1+x^2}}+C$

(28) $\displaystyle\int\sin^3x\,dx=\int\sin x\,(1-\cos^2x)\,dx=\int(\cos^2x-1)\,d\cos x=\frac{\cos^3x}{3}-\cos x+C$

(29) $\displaystyle\int\frac{dx}{\sin^2x\cos^2x}=\int\left(\frac{1}{\sin^2x}+\frac{1}{\cos^2x}\right)dx=\int(\sec^2x+\csc^2x)\,dx=\tan x-\cot x+C$

(30). $\displaystyle\int\frac{\sin x\cos^3x}{1+\cos^2x}\,dx=-\int\frac{\cos^3x\,d\cos x}{1+\cos^2x}=-\int\frac{\cos^3x+\cos x-\cos x}{\cos^2x+1}\,d\cos x$

$\displaystyle=-\int\cos x\,d\cos x+\int\frac{\cos x}{1+\cos^2x}\,d\cos x=-\frac{\cos^2x}{2}+C+\frac{1}{2}\int\frac{d\cos^2x}{1+\cos^2x}$

$\displaystyle=\frac{-\cos^2x}{2}+\frac{1}{2}\ln(1+\cos^2x)+C$

(31). $\displaystyle\int\frac{\sin^2x}{\cos^6x}\,dx=\int\frac{\sin^2x}{\cos^4x}\,d\tan x=\int\tan^2x\cdot\sec^2x\,d\tan x=\int\tan^2x(\tan^2x+1)\,d\tan x$

$\displaystyle=\int(\tan^4x+\tan^2x)\,d\tan x=\frac{\tan^5x}{5}+\frac{\tan^3x}{3}+C$

(32) $\displaystyle\int\frac{dx}{(1-x)^{\frac{3}{2}}}=-\int(1-x)^{-\frac{3}{2}}\,d(1-x)=\frac{2}{\sqrt{1-x}}+C$

(33) $\displaystyle\int\frac{x^2dx}{\sqrt{x^2+a^2}}=\int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx=\int\sqrt{x^2+a^2}\,dx-a^2\int\frac{dx}{\sqrt{x^2+a^2}}$

$\displaystyle=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|-a^2\ln\left|x+\sqrt{x^2+a^2}\right|+C=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|$

练习7-3.

1.(1) $\int x^2e^{-2x}dx=-\frac{1}{2}\int x^2de^{-2x}=-\frac{1}{2}(x^2e^{-2x}-\int 2xe^{-2x}dx)=-\frac{1}{2}[x^2e^{-2x}+\int xde^{-2x}]$

$=-\frac{1}{2}[x^2e^{-2x}+xe^{-2x}-\int e^{-2x}dx]=-\frac{1}{2}[x^2e^{-2x}+xe^{-2x}+\frac{1}{2}e^{-2x}]$

$=-\frac{1}{2}(x^2+x+\frac{1}{2})e^{-2x}+C$

(2) $\int\frac{x}{\sqrt{1+x^2}}\cdot\ln(x+\sqrt{1+x^2})dx=\int\ln(x+\sqrt{1+x^2})d\sqrt{1+x^2}=\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})-\int\sqrt{1+x^2}d\ln(x+\sqrt{1+x^2})$

$=\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})-\int\sqrt{1+x^2}\cdot\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}dx=\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})-x+C$

(3) $\int x^3e^{-x^2}dx=-\frac{1}{2}\int x^2(-2x\cdot e^{-x^2})dx=-\frac{1}{2}\int x^2de^{-x^2}=-\frac{1}{2}(x^2e^{-x^2}-\int e^{-x^2}dx^2)$

$=-\frac{1}{2}(x^2e^{-x^2}+e^{-x^2})=-\frac{1}{2}(x^2+1)e^{-x^2}+C$

(4) $\int e^{\sqrt{x}}dx=\int 2\sqrt{x}\cdot\frac{e^{\sqrt{x}}}{2\sqrt{x}}dx=\int 2\sqrt{x}de^{\sqrt{x}}=2\sqrt{x}e^{\sqrt{x}}-\int e^{\sqrt{x}}d(2\sqrt{x})=2\sqrt{x}e^{\sqrt{x}}-2\int e^{\sqrt{x}}d\sqrt{x}$

$=2(\sqrt{x}-1)e^{\sqrt{x}}+C.$

(5) $\int(\frac{\ln x}{x})^2dx=-\int\ln^2x\,d\frac{1}{x}=-(\frac{\ln^2x}{x}-\int\frac{1}{x}d\ln^2x)=-(\frac{\ln^2x}{x}-2\int\frac{\ln x}{x^2}dx)$

$=-[\frac{\ln^2x}{x}+2\int\ln x\,d\frac{1}{x}]=-[\frac{\ln^2x}{x}+2(\frac{\ln x}{x}-\int\frac{1}{x^2}dx)]=-\frac{\ln^2x+2\ln x+2}{x}+C.$

(6) $\int x^n \ln x\,dx = \frac{1}{n+1}\int \ln x \cdot dx^{n+1} = \frac{1}{n+1}\left(x^{n+1}\ln x - \int x^{n+1} d\ln x\right)$

$= \frac{1}{n+1}\left(x^{n+1}\ln x - \int x^n dx\right) = \frac{1}{n+1}\left(x^{n+1}\ln x - \frac{x^{n+1}}{n+1}\right) + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$

(7) $\int \sqrt{x}\ln^2 x\,dx = \frac{2}{3}\int \ln^2 x\,dx^{\frac{3}{2}} = \frac{2}{3}\left(\ln^2 x \cdot x^{\frac{3}{2}} - \int x^{\frac{3}{2}} d\ln^2 x\right) = \frac{2}{3}\left(x\sqrt{x}\ln^2 x - \int 2\sqrt{x}\ln x\,dx\right)$

$= \frac{2}{3}\left[x\sqrt{x}\ln^2 x - \frac{4}{3}\int \frac{3}{2}\sqrt{x}\ln x\,dx\right] = \frac{2}{3}\left[x\sqrt{x}\ln^2 x - \frac{4}{3}\int \ln x\,dx^{\frac{3}{2}}\right]$

$= \frac{2}{3}\left[x\sqrt{x}\ln^2 x - \frac{4}{3}\left(x\sqrt{x}\ln x - \int x^{\frac{1}{2}}dx\right)\right] = \frac{2}{3}\left[x\sqrt{x}\ln^2 x - \frac{4}{3}x\sqrt{x}\ln x + \frac{8}{9}x\sqrt{x}\right] + C$

$= \frac{2}{3}x\sqrt{x}\left(\ln^2 x - \frac{4}{3}\ln x + \frac{8}{9}\right) + C$

(8) $\int x^2 \sin 2x\,dx = -\frac{1}{2}\int x^2 d\cos 2x = -\frac{1}{2}\left(x^2\cos 2x - \int 2x\cos 2x\,dx\right)$

$= -\frac{1}{2}\left(x^2\cos 2x - \int x\,d\sin 2x\right) = -\frac{1}{2}\left[x^2\cos 2x - \left(x\sin 2x - \int \sin 2x\,dx\right)\right]$

$= -\frac{1}{2}\left[x^2\cos 2x - \left(x\sin 2x + \frac{1}{2}\cos 2x\right)\right] + C = \frac{x}{2}\sin 2x - \frac{2x^2-1}{4}\cos 2x + C$

(9) 令 $\int e^{2x}\sin^2 x\,dx = I$.

$\therefore I = \frac{1}{2}\int \sin^2 x\,de^{2x} = \frac{1}{2}\left(\sin^2 x\,e^{2x} - \int e^{2x} d\sin^2 x\right) = \frac{1}{2}\left(\sin^2 x\,e^{2x} - \frac{1}{2}\int \sin 2x\,de^{2x}\right)$

$= \frac{1}{2}\left[\sin^2 x\,e^{2x} - \frac{1}{2}\left(\sin 2x - \int e^{2x} d\sin 2x\right)\right] = \frac{1}{2}\left[\sin^2 x\,e^{2x} - \frac{1}{2}\left(\sin 2x - 2\int \cos 2x\,e^{2x}dx\right)\right]$

$= \frac{1}{2}\sin^2 x\,e^{2x} - \frac{1}{4}\sin 2x\,e^{2x} + \frac{1}{2}\int (1-2\sin^2 x)e^{2x}dx$

$= \frac{1}{2}\sin^2 x\,e^{2x} - \frac{1}{4}\sin 2x\,e^{2x} + \frac{1}{4}e^{2x} - I$ $\therefore I = \frac{e^{2x}}{8}\left(2\sin^2 x - \sin 2x + 1\right) + C$

(10) $\int \arctan x dx = x\arctan x - \int x \, d\arctan x = x\arctan x - \int \frac{x dx}{1+x^2}$

$= x\arctan x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = x\arctan x - \frac{1}{2}\ln(x^2+1) + C$

(11) $\int \frac{\arcsin x}{x^2} dx = -\int \arcsin x \, d\frac{1}{x} = -\left(\frac{\arcsin x}{x} - \int \frac{1}{x} d\arcsin x\right)$

$= -\frac{\arcsin x}{x} + \int \frac{1}{x\sqrt{1-x^2}} dx$

令 $x = \sin t$, $\sqrt{1-x^2} = \cos t$, $t \in (0, \frac{\pi}{2})$

$\therefore \int \frac{1}{x\sqrt{1-x^2}} dx = \int \frac{1}{\sin t \cos t} d\sin t = \int \frac{1}{\sin t} dt = \int \csc t \, dt = \ln|\csc t - \cot t| + C$

$= \ln\left|\frac{1-\cos t}{\sin t}\right| + C = \ln \frac{1-\sqrt{1-x^2}}{|x|} + C$

$\therefore \int \frac{\arcsin x}{x^2} dx = \ln \frac{1-\sqrt{1-x^2}}{|x|} - \frac{\arcsin x}{x} + C$

(12)

$\int x(\arctan x)^2 dx = \int (\arctan x)^2 . d\frac{x^2+1}{2} = \frac{x^2+1}{2}(\arctan x)^2 - \int \frac{x^2+1}{2} d(\arctan x)^2$

$= \frac{(x^2+1)(\arctan x)^2}{2} - \int \arctan x dx$

由(10). 原式 $= \frac{x^2+1}{2}(\arctan x)^2 - x\arctan x + \frac{1}{2}\ln(x^2+1) + C$

(13)

$$\int (\arcsin x)^2 dx = x(\arcsin x)^2 - \int \frac{2x\arcsin x}{\sqrt{1-x^2}} dx$$

$$= x(\arcsin x)^2 + \int 2\arcsin x \, d\sqrt{1-x^2} = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - \int 2\sqrt{1-x^2} \, d\arcsin x$$

$$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2\int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} dx = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

(14)

$$\int \sin x \ln \tan x \, dx = \int \sin x \ln \sin x \, dx - \int \sin x \ln \cos x \, dx = -\int \ln \sin x \, d\cos x + \int \ln \cos x \, d\cos x$$

由于: $\int \ln \sin x \, d\cos x = \cos x \ln \sin x - \int \cos x \, d\ln \sin x = \cos x \ln \sin x - \int \frac{1-\sin^2 x}{\sin x} dx$

$$= \cos x \ln \sin x - \int \csc x \, dx + \int \sin x \, dx = \cos x \ln \sin x - \ln|\csc x - \cot x| - \cos x + C$$

$$\int \ln \cos x \, d\cos x = \cos x \ln \cos x - \int \cos x \, d\ln \cos x = \cos x \ln \cos x + \int \sin x \, dx$$

$$= \cos x \ln \cos x - \cos x + C$$

$$\therefore \int \sin x \ln \tan x \, dx = \cos x \cdot \ln(\cot x) + \ln|\csc x - \cot x| + C$$

(15)

$$\int \ln(x+\sqrt{1+x^2}) dx = x\ln(x+\sqrt{1+x^2}) - \int x \, d\ln(x+\sqrt{1+x^2}) = x\ln(x+\sqrt{1+x^2}) - \int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

(16)

$$\int x \ln\frac{1+x}{1-x} dx = -\int \ln\frac{1+x}{1-x} \, d\frac{1-x^2}{2} = \frac{x^2-1}{2}\ln\frac{1+x}{1-x} + \int \frac{1-x^2}{2} \, d\ln\frac{1+x}{1-x}$$

$$= \frac{x^2-1}{2}\ln\frac{1+x}{1-x} + \int dx = \frac{x^2-1}{2}\ln\frac{1+x}{1-x} + x + C$$

(17)

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int x\,d\sin(\ln x) = x\sin(\ln x) - \int \cos(\ln x)dx$$

$$= x\sin(\ln x) - \left[x\cos(\ln x) - \int x\,d\cos(\ln x)\right] = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$\therefore \int \sin(\ln x)dx = \frac{x}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + C$$

(18) $$\int \frac{x\,dx}{\cos^2 x} = \int x\,d\tan x = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C$$

(19) $$\int \frac{\ln(\sin x)}{\sin^2 x}dx = -\int \ln(\sin x)\,d\cot x = -\cot x\ln(\sin x) + \int \cot x \cdot d\ln(\sin x)$$

$$= -\cot x\ln(\sin x) + \int \cot^2 x\,dx = -\cot x\ln(\sin x) + \int \frac{1-\sin^2 x}{\sin^2 x}dx$$

$$= -\cot x\ln(\sin x) + \int \csc^2 x\,dx - \int 1\,dx = -\cot x\ln(\sin x) - \cot x - x + C$$

(20) $$\int \frac{x^2}{\sqrt{x^2-2}}dx = \int \frac{x^2-2+2}{\sqrt{x^2-2}}dx = \int \sqrt{x^2-2}\,dx + \int \frac{2}{\sqrt{x^2-2}}dx$$

$$= x\sqrt{x^2-2} - \int \frac{x^2}{\sqrt{x^2-2}}dx + 2\int \frac{dx}{\sqrt{x^2-2}}$$

由 P218 例10, $\int \frac{dx}{\sqrt{x^2-2}} = \ln\left|x+\sqrt{x^2-2}\right| + C$

$$\therefore \int \frac{x^2}{\sqrt{x^2-2}}dx = \frac{x\sqrt{x^2-2}}{2} + \ln\left|x+\sqrt{x^2-2}\right| + C$$

(21) $\int\sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int\frac{x^2}{\sqrt{x^2+a^2}}\,dx = x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int\frac{dx}{\sqrt{x^2+a^2}}$

由P217例8: $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2})+C$

$\therefore \int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$

(22) $\int x^2\sqrt{x^2+a^2}\,dx = x^3\sqrt{x^2+a^2} - \int x\,d(x^2\sqrt{x^2+a^2}) = x^3\sqrt{x^2+a^2} - \int x\left(2x\sqrt{x^2+a^2} + \frac{x^3}{\sqrt{x^2+a}}\right)dx$

$= x^3\sqrt{x^2+a^2} - 2\int x^2\sqrt{x^2+a^2}\,dx - \int\frac{x^4}{\sqrt{x^2+a^2}}\,dx$

其中 $\int\frac{x^4}{\sqrt{x^2+a^2}}\,dx = \int\frac{x^4-a^4+a^4}{\sqrt{x^2+a^2}}\,dx = a^4\int\frac{dx}{\sqrt{x^2+a^2}} + \int\sqrt{x^2+a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2+a^2}} + \int\frac{x^2}{\sqrt{x^2+a^2}}\,dx$

$= (a^4-a^2)\int\frac{dx}{\sqrt{x^2+a^2}} + \int\sqrt{x^2+a^2}\,dx + \int\sqrt{x^2+a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2+a^2}} = (a^4-2a^2)\int\frac{dx}{\sqrt{x^2+a^2}} + 2\int\sqrt{x^2+a^2}\,dx$

由(21), $\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$.

由P217例8: $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2})+C$

$\therefore \int\frac{x^4}{\sqrt{x^2+a^2}}\,dx = (a^4-2a^2)\ln(x+\sqrt{x^2+a^2}) + x\sqrt{x^2+a^2} + a^2\ln(x+\sqrt{x^2+a^2})+C$

$\therefore \int x^2\sqrt{x^2+a^2}\,dx = \frac{(x^3-x)}{3}\sqrt{x^2+a^2} - \frac{(a^4-a^2)}{3}\ln(x+\sqrt{x^2+a^2})+C$

(23) $\int x^n e^{-x}\,dx = -\int x^n\,de^{-x} = -(x^n e^{-x} - \int e^{-x}\,dx^n) = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx$.

记 $I_n = \int x^n e^{-x}\,dx$ 则 $I_n = nI_{n-1} - x^n e^{-x}$

$\because I_1 = \int xe^{-x}\,dx = -\int x\,de^{-x} = -(xe^{-x} - \int e^{-x}\,dx) = -(x+1)e^{-x}$,

$I_2 = -(x^2+2x+2)e^{-x}$, $I_3 = -(x^3+3x^2+6x+6)e^{-x}$

$\therefore I_n = -e^{-x}\cdot n!\cdot\sum_{k=0}^{n}\frac{x^k}{k!}+C$ （可由数学归纳法证明，此处证略）

练习7.4.

1.(1)
$$\int \frac{x+1}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{dx}{x^2+x+1}$$
$$= \frac{1}{2}\ln(x^2+x+1) + C + \frac{1}{2}\int \frac{d(x+\frac{1}{2})}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} = \frac{1}{2}\ln(x^2+x+1) + \frac{2}{3}\sqrt{3}\arctan(\frac{2}{3}\sqrt{3}x+\frac{\sqrt{3}}{3}) + C$$

(2)
$$\int \frac{dx}{a+bx^2} = \frac{1}{b}\int \frac{dx}{x^2+(\frac{\sqrt{a}}{\sqrt{b}})^2} = \frac{1}{b}\cdot\frac{\sqrt{b}}{\sqrt{a}}\arctan\frac{\sqrt{b}}{\sqrt{a}}x + C = \frac{1}{\sqrt{ab}}\arctan\sqrt{\frac{b}{a}}x + C$$

(3)
$$\int \frac{x-1}{(x^2+2x+3)^2}dx = \frac{1}{2}\int \frac{(2x+2)dx}{(x^2+2x+3)^2} - 2\int \frac{dx}{(x^2+2x+3)^2} = \frac{1}{2}\int \frac{d(x^2+2x+3)}{(x^2+2x+3)^2} - 2\int \frac{dx}{(x^2+2x+3)^2}$$
$$= -\frac{1}{2}\cdot\frac{1}{x^2+2x+3} - \int \frac{2dx}{[(x+1)^2+2]^2} = -\frac{1}{2(x^2+2x+3)} - \int \frac{(x+1)^2+2-(x+1)^2}{[(x+1)^2+2]^2}dx$$
$$= -\frac{1}{2}\cdot\frac{1}{x^2+2x+3} - \int \frac{dx}{x^2+2x+3} + \int \frac{(x+1)^2dx}{[(x+1)^2+2]^2} = -\frac{1}{2(x^2+2x+3)} - \int \frac{dx}{x^2+2x+3} + \frac{1}{2}\int \frac{(x+1)d(x^2+2x+3)}{(x^2+2x+3)^2}$$
$$= -\frac{1}{2(x^2+2x+3)} - \int \frac{dx}{x^2+2x+3} - \frac{1}{2}\int (x+1)\,d\frac{1}{x^2+2x+3}$$
$$= -\frac{1}{2(x^2+2x+3)} - \int \frac{dx}{x^2+2x+3} - \frac{1}{2}\left[\frac{x+1}{x^2+2x+3} - \int \frac{dx}{x^2+2x+3}\right]$$
$$= -\frac{1}{2(x^2+2x+3)} - \frac{x+1}{2(x^2+2x+3)} - \frac{1}{2}\int \frac{dx}{x^2+2x+3}$$
$$= -\frac{1}{2(x^2+2x+3)} - \frac{x+1}{2(x^2+2x+3)} - \frac{\sqrt{2}}{4}\int \frac{d\frac{x+1}{\sqrt{2}}}{1+(\frac{x+1}{\sqrt{2}})^2}$$
$$= -\frac{\sqrt{2}}{4}\arctan\frac{x+1}{\sqrt{2}} - \frac{x+2}{2(x^2+2x+3)} + C$$

(4)

$$\int\frac{x^2}{(x^2+2x+2)^2}dx=\int\frac{x^2+2x+2-(2x+2)}{(x^2+2x+2)^2}dx=\int\frac{dx}{x^2+2x+2}-\int\frac{(2x+2)dx}{(x^2+2x+2)^2}$$

$$=\int\frac{d(x+1)}{1+(x+1)^2}-\int\frac{d(x^2+2x+2)}{(x^2+2x+2)^2}=\arctan(x+1)+\frac{1}{x^2+2x+2}+C$$

(5)

$$\int\frac{x^2-4x-2}{x(x^2+1)}dx=\int\frac{x^2+1-(4x+3)}{x(x^2+1)}dx=\int\frac{dx}{x}-\int\frac{3(x^2+1)-x(3x-4)}{x(x^2+1)}dx$$

$$=\int\frac{dx}{x}-\int\frac{3dx}{x}+\int\frac{3x-4}{x^2+1}dx$$

$$=-2\int\frac{dx}{x}+\frac{3}{2}\int\frac{2xdx}{x^2+1}-4\int\frac{dx}{1+x^2}=-2\int\frac{dx}{x}+\frac{3}{2}\int\frac{d(x^2+1)}{x^2+1}-4\int\frac{dx}{1+x^2}$$

$$=-2\ln|x|+\frac{3}{2}\ln(x^2+1)-4\arctan x+C$$

(6).

$$\int\frac{xdx}{(x+1)^2(x^2+1)}=\frac{1}{2}\int\left[\frac{1}{1+x^2}-\frac{1}{(x+1)^2}\right]dx=\frac{1}{2}\int\frac{dx}{1+x^2}-\frac{1}{2}\int\frac{d(x+1)}{(x+1)^2}$$

$$=\frac{1}{2}\arctan x+\frac{1}{2(x+1)}+C$$

(7)

$$\int\frac{xdx}{x^2-2x\cos\alpha+1}=\int\frac{xdx}{(x-\cos\alpha)^2+\sin^2\alpha}=\frac{1}{2}\int\frac{2x-2\cos\alpha+2\cos\alpha}{(x-\cos\alpha)^2+\sin^2\alpha}dx$$

$$=\frac{1}{2}\int\frac{d(x^2-2x\cos\alpha+1)}{x^2-2x\cos\alpha+1}+\int\frac{\cot\alpha}{\left(\frac{x-\cos\alpha}{\sin\alpha}\right)^2+1}d\frac{x-\cos\alpha}{\sin\alpha}$$

$$=\frac{1}{2}\ln(x^2-2x\cos\alpha+1)+\cot\alpha\arctan\frac{x-\cos\alpha}{\sin\alpha}+C$$

(8) $$\int \frac{x^5 dx}{x^6-x^3-2} = \frac{1}{3}\int \frac{x^3 dx^3}{(x^3-2)(x^3+1)} = \frac{1}{3}\int \frac{\frac{2}{3}(x^3+1)+\frac{1}{3}(x^3-2)}{(x^3-2)(x^3+1)} dx^3$$

$$= \frac{2}{9}\int \frac{dx^3}{x^3-2} + \frac{1}{9}\int \frac{dx^3}{x^3+1} = \frac{2}{9}\ln|x^3-2| + \frac{1}{9}\ln|x^3+1| + C$$

(9)

$$\int \frac{x^3+1}{x(x-1)^3}dx = \int \frac{x^2}{(x-1)^3}dx + \int \frac{dx}{x(x-1)^3} = \int \frac{(x+1)(x-1)}{(x-1)^3}dx + \int \frac{1}{(x-1)^3}dx + \int \frac{dx}{x(x-1)^3}$$

$$= \int \frac{x+1}{(x-1)^2}dx + \int \frac{dx}{(x-1)^3} + \int \frac{dx}{x(x-1)^3} = \int \frac{x-1+2}{(x-1)^2}dx + \int \frac{dx}{(x-1)^3} + \int \frac{dx}{x(x-1)^3}$$

$$= \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x-1)^3} + \int \frac{dx}{x(x-1)^3} = \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + \int \frac{dx}{x(x-1)^3}$$

令 $$\frac{1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} = \frac{(A+B)x^3-(3A+2B-C)x^2+(3A+B-C+D)x-A}{x(x-1)^3}.$$

有 $$\begin{cases} A+B=0 \\ -3A-2B+C=0 \\ 3A+B-C+D=0 \\ -A=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \\ C=-1 \\ D=1 \end{cases} \quad \therefore \int \frac{dx}{x(x-1)^3} = -\int \frac{dx}{x} + \int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x-1)^3}$$

$\therefore$ 原式 $$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} - \ln|x| + \ln|x-1| + \frac{1}{x-1} - \frac{1}{2(x-1)^2} + C$$

$$= 2\ln|x-1| - \ln|x| - \frac{1}{x-1} - \frac{1}{(x-1)^2} + C$$

(10) $$\int \frac{dx}{(x+1)(x+2)^2(x+3)^3} = \int \left[\frac{1}{8(x+1)} + \frac{2}{x+2} - \frac{1}{(x+2)^2} - \frac{17}{8(x+3)} - \frac{5}{4(x+3)^2} - \frac{1}{2(x+3)^3}\right]dx$$

$$= \frac{1}{8}\ln|x+1| + 2\ln|x+2| + \frac{1}{x+2} - \frac{17}{8}\ln|x+3| + \frac{5}{4(x+3)} + \frac{1}{4(x+3)^2} + C$$

(11)

解: $$\int \frac{x^{3n-1}}{x^n+1}dx = \int \frac{x^{n-1}\cdot x^{2n}dx}{x^n+1} = \frac{1}{n}\int \frac{x^{2n}dx^n}{x^n+1} = \frac{1}{n}\int \frac{x^{2n}-1+1}{x^n+1}dx$$

$$= \frac{1}{n}\int (x^n-1)dx + \frac{1}{n}\int \frac{dx}{x^n+1} = \frac{1}{2n}x^{2n} - \frac{1}{n}x^n + \frac{1}{n}\ln|x^n+1| + C$$

练习7-5

(1) $\int \frac{dx}{\sin x\cos^4 x} = \int \frac{\sin x\,dx}{(1-\cos^2 x)\cos^4 x} = -\int \frac{d\cos x}{(1-\cos^2 x)\cos^4 x}$

$= -\left(\int \frac{d\cos x}{1-\cos^2 x} + \int \frac{1+\cos^2 x}{\cos^4 x} d\cos x\right) = \int \frac{dx}{\sin x} - \int \frac{d\cos x}{\cos^4 x} - \int \frac{d\cos x}{\cos^2 x}$

$= \ln|\csc x - \cot x| + \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + C$

(2) $\int \sin^4 x \cos^5 x\,dx = \int \sin^4 x\cos^4 x\,d\sin x = \int \sin^4 x\,(1-\sin^2 x)^2 d\sin x$

$= \int \sin^8 x\,d\sin x - 2\int \sin^6 x\,d\sin x + \int \sin^4 x\,d\sin x$

$= \frac{1}{9}\sin^9 x - \frac{2}{7}\sin^7 x + \frac{1}{5}\sin^5 x + C$

(3)

令 $I = \int \frac{dx}{\sin^3 x} = \int \csc^3 x\,dx = -\int \csc x\,d\cot x = -\csc x\cot x + \int \cot x\,d\csc x$

$= -\csc x\cot x - \int \cot^2 x\csc x\,dx = -\csc x\cot x - \int (\csc^2 x - 1)\csc x\,dx$

$= -\csc\cot x + \int \csc x\,dx - \int \csc^3 x\,dx = -\csc x\cot x + \ln|\csc x - \cot x| - I$

$\therefore I = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$

(4) $\int \frac{dx}{\sin^4 x\cos^4 x} = \int \frac{(\sin^2 x+\cos^2 x)^2}{\sin^4 x\cos^4 x}dx = \int \frac{dx}{\sin^4 x} + \int \frac{dx}{\cos^4 x} + 2\int \frac{dx}{\sin^2\cos^2 x}$

$= \int \frac{\sin^2 x+\cos^2 x}{\sin^4 x}dx + \int \frac{\sin^2 x+\cos^2 x}{\cos^4 x}dx + 2\int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx$

$= 3\int \csc^2 x\,dx + 3\int \sec^2 x\,dx + \int \cot^2 x\cdot\csc^2 x\,dx + \int \tan^2 x\sec^2 x\,dx$

$= 3\int \csc^2 x\,dx + 3\int \sec^2 x\,dx - \int \cot^2 x\,d\cot x + \int \tan^2 x\,d\tan x$

$= 3\tan x - 3\cot x - \frac{1}{3}\cot^3 x + \frac{1}{3}\tan^3 x + C$

(5) $\int \sin 5x\cos 3x\,dx = \int (\frac{1}{2}\sin 8x + \frac{1}{2}\sin 2x)dx = \frac{1}{2}\int \sin 8x\,dx + \frac{1}{2}\int \sin 2x\,dx$

$= -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$

(6) $\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{d\cos x}{(\cos x+2)(\cos^2 x-1)} = \int \frac{dx}{(\cos x+2)(\cos x+1)(\cos x-1)}$

$= \frac{1}{3}\int \frac{d\cos x}{\cos x+2} - \frac{1}{2}\int \frac{d\cos x}{\cos x+1} + \frac{1}{6}\int \frac{d\cos x}{\cos x-1}$

$= \frac{1}{3}\ln(\cos x+2) - \frac{1}{2}\ln(\cos x+1) + \frac{1}{6}\ln(1-\cos x) + C$

(7) 令 $\tan\frac{x}{2}=t$. 则 $x=2\arctan t$

$$\therefore \int\frac{dx}{2\sin x-\cos x+5}=\int\frac{2d\arctan t}{\frac{4t}{1+t^2}-\frac{1-t^2}{1+t^2}+\frac{5+5t^2}{1+t^2}}=\int\frac{\frac{2}{1+t^2}dt}{\frac{4t}{1+t^2}-\frac{1-t^2}{1+t^2}+\frac{5+5t^2}{1+t^2}}$$

$$=\int\frac{dt}{3t^2+2t+2}=\int\frac{dt}{3(t+\frac{1}{3})^2+\frac{5}{3}}=\frac{1}{\sqrt5}\int\frac{d(\frac{3}{\sqrt5}t+\frac{1}{\sqrt5})}{1+(\frac{3}{\sqrt5}t+\frac{1}{\sqrt5})^2}=\frac{1}{\sqrt5}\arctan(\frac{3}{\sqrt5}t+\frac{1}{\sqrt5})+C$$

$$=\frac{\sqrt5}{5}\arctan(\frac{\sqrt5}{5}+\frac{3}{5}\sqrt5\tan\frac{x}{2})+C$$

(8) $$\int\frac{\sin x\cos x}{1+\sin^4 x}dx=\int\frac{\sin x\, d\sin x}{1+\sin^4 x}=\frac{1}{2}\int\frac{d\sin^2 x}{1+(\sin^2 x)^2}=\frac{1}{2}\arctan(\sin^2 x)+C$$

(9) 令 $\tan\frac{x}{2}=t$ 则 $x=2\arctan t$

$$\therefore \int\frac{dx}{1+\varepsilon\cos x}=\int\frac{2d\arctan t}{1+\frac{(1-t^2)\varepsilon}{1+t^2}}=\int\frac{\frac{2}{1+t^2}dt}{1+\frac{(1-t^2)}{1+t^2}\varepsilon}=\int\frac{2dt}{1+t^2+\varepsilon-\varepsilon t^2}=\int\frac{\frac{2}{1+\varepsilon}dt}{1+\frac{1-\varepsilon}{1+\varepsilon}t^2}$$

① 当 $0<\varepsilon<1$ 时 $$\int\frac{dx}{1+\varepsilon\cos x}=\int\frac{\frac{2}{\sqrt{1-\varepsilon^2}}d\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}t}{1+\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}t\right)^2}=\frac{2}{\sqrt{1-\varepsilon^2}}\arctan\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}t+C$$

$$=\frac{2}{\sqrt{1-\varepsilon^2}}\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{x}{2}\right)+C$$

② 当 $\varepsilon>1$ 时 $$\int\frac{dx}{1+\varepsilon\cos x}=\frac{1}{1+\varepsilon}\int\left(\frac{1}{1+\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t}+\frac{1}{1-\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t}\right)dt=\frac{1}{1+\varepsilon}\int\frac{dt}{1+\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t}+\frac{1}{1+\varepsilon}\int\frac{dt}{1-\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t}$$

$$=\frac{1}{\sqrt{\varepsilon^2-1}}\int\frac{d(1+\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t)}{1+\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t}-\frac{1}{\sqrt{\varepsilon^2-1}}\int\frac{d(1-\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t)}{1-\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t}=\frac{1}{\sqrt{\varepsilon^2-1}}\ln\left|1+\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t\right|-\frac{1}{\sqrt{\varepsilon^2-1}}\ln\left|1-\sqrt{\frac{\varepsilon-1}{\varepsilon+1}}t\right|+C$$

$$=\frac{1}{\sqrt{\varepsilon^2-1}}\ln\left|\frac{\sqrt{\varepsilon+1}+\sqrt{\varepsilon-1}\,t}{\sqrt{\varepsilon+1}-\sqrt{\varepsilon-1}\,t}\right|+C$$

(10)

$$\int\frac{dx}{a\sin x+b\cos x}=\frac{1}{\sqrt{a^2+b^2}}\int\frac{dx}{\sin(x+\varphi)}=\frac{1}{\sqrt{a^2+b^2}}\int\csc(x+\varphi)\,d(x+\varphi)$$

$$=\frac{1}{\sqrt{a^2+b^2}}\ln|\csc(x+\varphi)-\cot(x+\varphi)|+C$$ (其中 $\tan\varphi=\frac{b}{a}$)

2.

(1)

$I_0 = x + C$

$I_1 = \int \sin x dx = -\cos x + C$

$n \geq 2$ 时 $I_n = \int \sin^n x dx = \int \sin^{n-2} x (1-\cos^2 x) dx = I_{n-2} - \int \sin^{n-2} x \cos^2 x dx$

$= I_{n-2} - \frac{1}{n-1} \int \cos x d \sin^{n-1} x$

$= I_{n-2} - \frac{1}{n-1} \left(\cos x \sin^{n-1} x + \int \sin^n x dx \right) = I_{n-2} - \frac{\cos x \sin^{n-1} x}{n-1} - \frac{1}{n-1} I_n$

$\therefore I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n} \cos x \sin^{n-1} x \quad (n \geq 2)$

(2) $I_0 = x + C \qquad I_1 = \int \cos x dx = \sin x + C$

$n \geq 2$ 时 $I_n = \int \cos^n x dx = \int \cos^{n-1} x d \sin x = \sin x \cos^{n-1} x - \int \sin x d \cos^{n-1} x$

$= \sin x \cos^{n-1} x + \int (n-1) \cos^{n-2} x \sin^2 x dx = \sin x \cos^{n-1} x + \int (n-1)(1-\cos^2 x) \cos^{n-2} x dx$

$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n} \sin x \cos^{n-1} x \quad (n \geq 2)$

(3) $K_0 = x + C \qquad K_1 = \int \csc x\,dx = \ln|\csc x - \cot x| + C.$

$n \geq 2$时 $K_{n-1} = \int \frac{\sin x}{\sin^n x}dx = -\int \frac{d\cos x}{\sin^n x} = -\left(\frac{\cos x}{\sin^n x} - \int \cos x\, d\frac{1}{\sin^n x}\right)$

$$= -n\int \frac{\cos^2 x}{\sin^{n+1}x}dx - \frac{\cos x}{\sin^n x} = -n\int \frac{1-\sin^2 x}{\sin^{n+1}x}dx - \frac{\cos x}{\sin^n x}$$

$$= nK_{n-1} - nK_{n+1} - \frac{\cos x}{\sin^n x}$$

$\therefore K_{n+1} = \frac{n-1}{n}K_{n-1} - \frac{\cos x}{n\sin^n x}$

即 $K_n = \frac{n-2}{n-1}K_{n-2} - \frac{\cos x}{(n-1)\sin^n x}.$

(4)

$L_0 = x + C \qquad L_1 = \int \sec x\,dx = \ln|\sec x + \tan x| + C.$

$n \geq 2$时 $L_{n-2} = \int \frac{\cos x\,dx}{\cos^{n-1}x} = \int \frac{d\sin x}{\cos^{n-1}x} = \frac{\sin x}{\cos^{n-1}x} - \int \sin x\, d\frac{1}{\cos^{n-1}x}$

$$= \frac{\sin x}{\cos^{n-1}x} - (n-1)\int \frac{\sin^2 x\,dx}{\cos^n x} = \frac{\sin x}{\cos^{n-1}x} - (n-1)\int \frac{1-\cos^2 x}{\cos^n x}dx$$

$$= \frac{\sin x}{\cos^{n-1}x} - (n-1)L_n + (n-1)L_{n-2}$$

$\therefore L_n = \frac{n-2}{n-1}L_{n-2} + \frac{\sin x}{\cos^{n-1}x}$

练习7.6

(1) 令 $\sqrt[6]{x}=t$，则 $x=t^6$

$$\therefore \int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})} = \int \frac{dt^6}{t^6(1+2t^3+t^2)} = \int \frac{6dt}{t(t+1)(2t^2-t+1)}$$

$$= \int \frac{dt}{t} - \frac{1}{4}\int \frac{dt}{t+1} - \frac{1}{4}\int \frac{6t-1}{2t^2-t+1}dt = \int \frac{dt}{t} - \frac{1}{4}\int \frac{d(t+1)}{t+1} - \frac{3}{8}\int \frac{(4t-1)dt}{2t^2-t+1} + \frac{3}{8}\int \frac{dt}{2t^2-t+1}$$

$$= \ln|t| - \frac{1}{4}\ln|t+1| - \frac{3}{8}\ln|2t^2-t+1| + \frac{3}{4\sqrt{7}}\arctan\left(\frac{4}{\sqrt{7}}t - \frac{1}{\sqrt{7}}\right) + C$$

$$= \ln|\sqrt[6]{x}| - \frac{1}{4}\ln|\sqrt[6]{x}+1| - \frac{3}{8}\ln|2\sqrt[3]{x} - \sqrt[6]{x}+1| + \frac{3}{4\sqrt{7}}\arctan\left(\frac{4}{\sqrt{7}}\sqrt[6]{x} - \frac{1}{\sqrt{7}}\right) + C$$

(2)
$$\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx = \int \frac{2x-2\sqrt{x^2-1}}{2}dx = \int x\,dx - \int \sqrt{x^2-1}\,dx = \frac{x^2}{2} + \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}| + C$$

(3) 令 $t=\sqrt[3]{\frac{x+1}{x-1}}$ 其中 $x=\frac{t^3+1}{t^3-1}$

$$\therefore \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{dx}{(x+1)^2\cdot\sqrt[3]{\frac{(x-1)^4}{(x+1)^4}}}$$ 令 $t=\sqrt[3]{\frac{x-1}{x+1}}$，即 $x=\frac{1+t^3}{1-t^3}$

$$\text{原式} = \int \frac{d\frac{1+t^3}{1-t^3}}{\frac{4}{(1-t^3)^2}\cdot t^4} = \int \frac{\frac{3t^2(1-t^3)+3t^2(1+t^3)}{(1-t^3)^2}dt}{\frac{4t^4}{(1-t^3)^2}} = \int \frac{6t^2dt}{4t^4} = \frac{3}{2}\int \frac{dt}{t^2} = -\frac{3}{2t} + C$$

$$= -\frac{3}{2}\sqrt[3]{\frac{x-1}{x+1}} + C$$

(4)

解：令 $\sqrt{x^2+x+1}=x-t$ $\therefore x^2+x+1=x^2-2tx+t^2$ $\therefore x=\frac{t^2-1}{2t+1}$

$\therefore \int\frac{dx}{x\sqrt{x^2+x+1}}=\int\frac{d\frac{t^2-1}{2t+1}}{\frac{t^2-1}{2t+1}\left(\frac{t^2-1}{2t+1}-t\right)}=\int\frac{\frac{2t(2t+1)-2(t^2-1)}{(2t+1)^2}dt}{\frac{t^2-1}{2t+1}\cdot\frac{-t^2-t-1}{2t+1}}$

$=\int\frac{(2t^2+2t+2)dt}{(1-t^2)(t^2+t+1)}=\int\frac{2dt}{1-t^2}=\int\frac{dt}{1-t}+\int\frac{dt}{1+t}=\ln|t+1|-\ln|t-1|+C$

$=\ln\left|\frac{t+1}{t-1}\right|+C=\ln\left|\frac{x-\sqrt{x^2+x+1}+1}{x-\sqrt{x^2+x+1}-1}\right|+C$

$=\ln\left|\frac{x+2-2\sqrt{x^2+x+1}}{3x}\right|+C$

(5) 令 $\sqrt{x^2-1}=x-t$ $\therefore x^2-1=x^2-2tx+t^2$ $\therefore x=\frac{t^2+1}{2t}=\frac{t}{2}+\frac{1}{2t}$

$\therefore \int\frac{dx}{(1+x^2)\sqrt{x^2-1}}=\int\frac{(\frac{1}{2}-\frac{1}{2t^2})dt}{(\frac{1}{2t}-\frac{t}{2})\frac{t^4+6t^2+1}{4t^2}}=\int\frac{\frac{t^2-1}{2t^2}dt}{\frac{1-t^2}{2t}\cdot\frac{t^4+6t^2+1}{4t^2}}=\int\frac{-dt}{\frac{t^4+6t^2+1}{4t}}$

$=\int\frac{-4tdt}{(t^2+3)^2-8}=-2\int\frac{d(t^2+3)}{(t^2+3)^2-8}=\frac{\sqrt{2}}{4}\int\left(\frac{1}{t^2+3+2\sqrt{2}}-\frac{1}{t^2+3-2\sqrt{2}}\right)d(t^2+3)$

$=\frac{\sqrt{2}}{4}\ln\left|\frac{t^2+3+2\sqrt{2}}{t^2+3-2\sqrt{2}}\right|+C=\frac{\sqrt{2}}{4}\ln\left|\frac{2x^2-2x\sqrt{x^2-1}+2\sqrt{2}+2}{2x^2-2x\sqrt{x^2-1}-2\sqrt{2}+2}\right|+C$

(6) $\int \frac{dx}{x^3\sqrt{x^2+1}} = \int \frac{x\,dx}{x^4\sqrt{x^2+1}} = \frac{1}{2}\int \frac{dx^2}{x^4\sqrt{x^2+1}}$ 令 $\sqrt{x^2+1}=t$ 即 $x^2=t^2-1$

$$\therefore \int \frac{dx}{x^3\sqrt{x^2+1}} = \frac{1}{2}\int \frac{d(t^2-1)}{t(t^2-1)^2} = \frac{1}{2}\cdot\int \frac{2dt}{(t^2-1)^2} = \int \frac{dt}{(t+1)^2(t-1)^2}$$

$$= \frac{1}{4}\int \frac{dt}{t+1} - \frac{1}{4}\int \frac{dt}{t-1} + \frac{1}{4}\int \frac{dt}{(t+1)^2} + \frac{1}{4}\int \frac{dt}{(t-1)^2} = \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{4(t-1)} - \frac{1}{4(t+1)} + C$$

$$= \frac{1}{4}\ln\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}-1} - \frac{\sqrt{x^2+1}}{2x^2} + C$$

(7) 令 $x=\sec t$ 即 $t=\arccos\frac{1}{x}$, $t=\arcsin\sqrt{1-\frac{1}{x^2}}$

$$\therefore \int \frac{dx}{(1+x^2)\sqrt{x^2-1}} = \int \frac{d\sec t}{\tan t\,(1+\sec^2 t)} = \int \frac{\sec t\,dt}{1+\sec^2 t} = \int \frac{\cos t\,dt}{1+\cos^2 t}$$

$$= \int \frac{d\sin t}{2-\sin^2 t} = \frac{1}{2\sqrt{2}}\int \frac{d\sin t}{\sqrt{2}-\sin t} + \frac{1}{2\sqrt{2}}\int \frac{d\sin t}{\sqrt{2}+\sin t} = \frac{\sqrt{2}}{4}\ln\frac{\sqrt{2}+\sin t}{\sqrt{2}-\sin t} + C$$

$$= \frac{\sqrt{2}}{4}\ln\frac{\sqrt{2}+\sqrt{1-\frac{1}{x^2}}}{\sqrt{2}-\sqrt{1-\frac{1}{x^2}}} + C = \frac{\sqrt{2}}{4}\ln\frac{\sqrt{2x^2}+\sqrt{x^2-1}}{\sqrt{2x^2}-\sqrt{x^2-1}} + C$$

(8) $\int \frac{x\,dx}{\sqrt[4]{x^3(a-x)}} = \int \sqrt[4]{\frac{x}{a-x}}\,dx$ 令 $\sqrt[4]{\frac{x}{a-x}}=t$, 即 $t^4=\frac{x}{a-x}$, $x=\frac{at^4}{1+t^4}$

$$\therefore \int \frac{x\,dx}{\sqrt[4]{x^3(a-x)}} = \int t\,d\frac{at^4}{1+t^4} = a\int \frac{4t^4\,dt}{(t^4+1)^2} = a\int \frac{t\,dt^4}{(t^4+1)^2} = a\int t\,d\frac{-1}{t^4+1}$$

$$= a\left(\frac{-t}{t^4+1} + \int \frac{dt}{t^4+1}\right) = \frac{-at}{t^4+1} + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}t-1) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}t+1) + C$$ (由P231例2)

$$= -\sqrt[4]{x(a-x)^3} + \frac{\sqrt{2}}{4}\left[\arctan\left(\sqrt[4]{\frac{4x}{a-x}}-1\right) + \arctan\left(\sqrt[4]{\frac{4x}{a-x}}+1\right)\right] + C$$

习题7.

1、(1) $\because f'(x)\cdot f(x)=\frac{1}{2}(f^2(x))'=1$ $\therefore (f^2)'=2$

$\therefore f^2(x)=2x+C$ $\therefore f(x)=\sqrt{2x+C}$ (C为常数)

(2) $f'(x)=\sqrt{1-(1-2\sin^2 x)}=\sqrt{2}|\sin x|$

$\therefore f(x)=-\sqrt{2}\cos x\cdot \mathrm{sgn}(\sin x)+C$ $\because f(\frac{\pi}{2})=1$

$\therefore f(\frac{\pi}{2})=C=1$ $\therefore f(x)=1-\sqrt{2}\cos x\cdot \mathrm{sgn}(\sin x)$

习题7 2.

(1) $\int\frac{x^2+1}{x^4+1}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int\frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2}=\int\frac{\frac{1}{\sqrt2}d(\frac{x}{\sqrt2}-\frac{1}{\sqrt2 x})}{1+(\frac{x}{\sqrt2}-\frac{1}{\sqrt2 x})^2}=\frac{\sqrt2}{2}\arctan\frac{\sqrt2}{2}(x-\frac{1}{x})+C$

(2) $\int\frac{x^2-1}{x^4+1}dx=\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}=\frac{1}{2\sqrt2}\int\left(\frac{1}{x+\frac{1}{x}-\sqrt2}-\frac{1}{x+\frac{1}{x}+\sqrt2}\right)d(x+\frac{1}{x})$

$=\frac{1}{2\sqrt2}\ln\left|\frac{x+\frac{1}{x}-\sqrt2}{x+\frac{1}{x}+\sqrt2}\right|+C=\frac{\sqrt2}{4}\ln\left|\frac{x^2-\sqrt2x+1}{x^2+\sqrt2x+1}\right|+C$

(3) $\int\frac{x^4+1}{x^6+1}dx=\int\frac{(x^4-x^2+1)+x^2}{(x^2+1)(x^4-x^2+1)}dx=\int\frac{dx}{1+x^2}+\int\frac{x^2dx}{1+x^6}=\arctan x+\frac{1}{3}\arctan x^3+C$

(4) $\int\frac{2^x\cdot3^x}{9^x-4^x}dx=\int\frac{2^x\cdot3^x dx}{(3^x+2^x)(3^x-2^x)}=\frac{1}{2}\int\frac{3^xdx}{3^x-2^x}-\frac{1}{2}\int\frac{3^xdx}{3^x+2^x}=\frac{1}{2}\int\frac{(\frac{3}{2})^xdx}{(\frac{3}{2})^x-1}-\frac{1}{2}\int\frac{(\frac{3}{2})^xdx}{(\frac{3}{2})^x+1}$

$=\frac{1}{2\ln\frac{3}{2}}\int\frac{d(\frac{3}{2})^x}{(\frac{3}{2})^x-1}-\frac{1}{2\ln\frac{3}{2}}\int\frac{d(\frac{3}{2})^x}{(\frac{3}{2})^x+1}=\frac{1}{2\ln\frac{3}{2}}\ln\left|\frac{(\frac{3}{2})^x-1}{(\frac{3}{2})^x+1}\right|+C=\frac{1}{2\ln\frac{3}{2}}\ln\left|\frac{3^x-2^x}{3^x+2^x}\right|+C$

(5) $\int\frac{\sqrt{x-a}}{\sqrt{x+a}}dx=\int\frac{x-a}{\sqrt{x^2-a^2}}dx=\int\frac{xdx}{\sqrt{x^2-a^2}}-a\int\frac{dx}{\sqrt{x^2-a^2}}=\sqrt{x^2-a^2}-a\ln|x+\sqrt{x^2-a^2}|+C$

(6) 令 $\sqrt{(x+a)(x+b)}=t(x+a)$ $\therefore t^2=\frac{x+b}{x+a}$ $\therefore x=\frac{at^2-b}{1-t^2}$

$\therefore \int\frac{dx}{\sqrt{(x+a)(x+b)}}=\int\frac{d\frac{at^2-b}{1-t^2}}{t\cdot\frac{at^2-b+a-at^2}{1-t^2}}=\int\frac{\frac{2at(1-t^2)+2t(at^2-b)}{(1-t^2)^2}dt}{t\cdot\frac{a-b}{1-t^2}}$

$=\int\frac{\frac{2\cancel{t}(\cancel{a-b})}{(1-t^2)^{\cancel{2}}}dt}{\cancel{t}\cdot\frac{\cancel{a-b}}{\cancel{1-t^2}}}=\int\frac{2dt}{1-t^2}=\int\frac{dt}{1+t}+\int\frac{dt}{1-t}$

$=\ln\left|\frac{1+t}{1-t}\right|+C=\ln\left|\frac{\sqrt{x+a}+\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\right|+C$

3.

(1) 令 $t=\arctan x$，则 $x=\tan t$

$\therefore \int \frac{x e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx = \int \frac{\tan t\, e^t\, d\tan t}{(1+x^2)^{\frac{3}{2}}} = \int e^t \cdot \frac{\sin t}{\cos t}\cdot \cos^3 t \cdot \frac{dt}{\cos^2 t}$

$= \int \sin t\, e^t dt = \int \sin t\, de^t = e^t \sin t - \int e^t \cos t\, dt = e^t \sin t - (e^t \cos t + \int \sin t\, e^t dt)$.

$\therefore \int \sin t\, e^t dt = \frac{e^t \sin t - e^t \cos t}{2}$

$\therefore$ 原式 $= \frac{e^{\arctan x}}{2}\cdot[\sin(\arctan x) - \cos(\arctan x)] + C$

(2) $\int \frac{xe^x}{(x+1)^2}dx = -\int xe^x d\frac{1}{x+1} = -\left(\frac{xe^x}{x+1} - \int \frac{1}{x+1}\cdot(x+1)e^x dx\right)$

$= e^x - \frac{xe^x}{x+1} + C = \frac{e^x}{x+1} + C$

(3)

令 $I = \int xe^x \cos x\, dx$

$\therefore I = \int xe^x d\sin x = xe^x \sin x - \int (x+1)e^x \sin x\, dx = xe^x \sin x + \int (x+1)e^x d\cos x$

$= xe^x \sin x + (x+1)e^x \cos x - \int (x+2)e^x \cos x\, dx = e^x[x\sin x + (x+1)\cos x] - I - 2\int e^x \cos x\, dx$

其中 $\int e^x \cos x\, dx = \int \cos x\, de^x = e^x \cos x + \int \sin x \cdot e^x dx = e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$

$\therefore 2\int e^x \cos x\, dx = e^x \cos x + e^x \sin x$ $\quad\therefore I = \frac{e^x}{2}[(x-1)\sin x + x\cos x] + C$

(4) $\int (1+x-\frac{1}{x})e^{x+\frac{1}{x}}dx = \int e^{x+\frac{1}{x}}dx + \int x(1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx = \int e^{x+\frac{1}{x}}dx + \int xe^{x+\frac{1}{x}}d(x+\frac{1}{x})$

$= \int e^{x+\frac{1}{x}}dx + \int x\, de^{x+\frac{1}{x}} = xe^{x+\frac{1}{x}} + C$

4 使用数学归纳法容易证明.

5. 令 $t=\frac{x+a}{x+b}$ $\qquad \therefore x=-b+\frac{a-b}{t-1}$

$$\therefore \int \frac{dx}{(x+a)^m(x+b)^n}=\int \frac{dx}{\left(\frac{x+a}{x+b}\right)^m\cdot(x+b)^{n+m}}=\int \frac{-\frac{a-b}{(t-1)^2}dt}{t^m\cdot\left(\frac{a-b}{t-1}\right)^{m+n}}$$

$$=-\frac{1}{(a-b)^{m+n-1}}\int \frac{(t-1)^{m+n-2}}{t^m}dt$$

$$=-\frac{1}{(a-b)^{m+n-1}}\cdot\int \sum_{k=0}^{m+n-2} C_{m+n-2}^{k}\, t^{n-k-2}(-1)^k dt$$

$$=-\frac{1}{(a-b)^{m+n-1}}\left[(-1)^{n-1}C_{m+n-2}^{n-1}\ln|t|+\sum_{k=0\text{且}k\neq n-1}^{m+n-2}\frac{(-1)^k C_{m+n-2}^{k}}{n-k-1}t^{n-k-1}\right]+C$$

$$=-\frac{1}{(a-b)^{m+n-1}}\left[(-1)^{n-1}C_{m+n-2}^{n-1}\ln\left|\frac{x+a}{x+b}\right|+\sum_{k=0\text{且}k\neq n-1}^{m+n-2}\frac{(-1)^k C_{m+n-2}^{k}}{n-k-1}\left(\frac{x+a}{x+b}\right)^{n-k-1}\right]+C$$

6.

(1)

解：由积化和差公式.

$$\int \cos x \cos 2x \cos 3x\,dx = \frac{1}{2}\int (\cos 3x + \cos x)\cos 3x\,dx$$

$$= \frac{1}{4}\int (\cos 6x + 1 + \cos 4x + \cos 2x)\,dx = \frac{1}{24}\sin 6x + \frac{1}{16}\sin 4x + \frac{1}{8}\sin 2x + \frac{x}{4} + C$$

(2)

解：
$$\int \frac{dx}{\sin x - \sin a} = \int \frac{dx}{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}} = \frac{1}{\cos a}\int \frac{\cos\left(\frac{x+a}{2} - \frac{x-a}{2}\right)dx}{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}$$

$$= \frac{1}{\cos a}\int \cot\frac{x-a}{2}\,d\frac{x-a}{2} + \frac{1}{\cos a}\int \tan\frac{x+a}{2}\,d\frac{x+a}{2}$$

$$= \frac{1}{\cos a}\ln\left|\sin\frac{x-a}{2}\right| - \frac{1}{\cos a}\ln\left|\cos\frac{x+a}{2}\right| + C$$

$$= \frac{1}{\cos a}\ln\left|\frac{\sin\frac{x-a}{2}}{\cos\frac{x+a}{2}}\right| + C$$

6.

(3)

$$\int \frac{dx}{\sin(x+a)\sin(x+b)} = \frac{1}{\sin(a-b)}\int \frac{\sin(a+x-b-x)\,dx}{\sin(x+a)\sin(x+b)}$$

$$= \frac{1}{\sin(a-b)}\int \frac{\sin(x+a)\cos(x+b)-\sin(x+b)\cos(x+a)}{\sin(x+a)\sin(x+b)}dx$$

$$= \frac{1}{\sin(a-b)}\int \cot(x+b)\,dx - \frac{1}{\sin(a-b)}\int \cot(x+a)\,dx$$

$$= \frac{1}{\sin(a-b)}\ln|\sin(x+b)| - \frac{1}{\sin(a-b)}\ln|\sin(x+a)| + C$$

$$= \frac{1}{\sin(a-b)}\cdot\ln\left|\frac{\sin(x+b)}{\sin(x+a)}\right| + C$$

(4) 令 $\sqrt{\tan x} = t$ $\therefore x = \arctan t^2$

$$\therefore \int\frac{dx}{\sqrt{\tan x}} = \int \frac{1}{t}\cdot\frac{2t}{1+t^4}dt = 2\int\frac{dt}{1+t^4}$$

$$= \frac{\sqrt{2}}{4}\ln\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} + \frac{\sqrt{2}}{2}\left[\arctan(\sqrt{2}t-1)+\arctan(\sqrt{2}t+1)\right]+C$$ (由P232 (3)/2)

$$= \frac{\sqrt{2}}{4}\ln\frac{\tan x+1+\sqrt{2\tan x}}{\tan x+1-\sqrt{2\tan x}} + \frac{\sqrt{2}}{2}\left[\arctan(\sqrt{2\tan x}-1)+\arctan(\sqrt{2\tan x}+1)\right]+C$$

(5)

$$\int\frac{dx}{\sqrt{\sin^3 x\cos^5 x}} = \int\frac{\sin^2 x+\cos^2 x}{\sqrt{\sin^3 x\cos^5 x}}dx = \int\frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x}\cdot\cos^2 x} + \int\frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x}\,\sin^2 x}$$

$$= \int\sqrt{\tan x}\,d\tan x - \int\frac{1}{\sqrt{\cot x}}d\cot x$$

$$= \frac{2}{3}\tan x\sqrt{\tan x} - 2\sqrt{\cot x} + C$$

南開大學 作业纸

系别________ 班级________ 姓名__________ 第 页

7. (1) $\int\sqrt{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int\sqrt{x^2+\frac{1}{x^2}}\,d2x = \frac{1}{2}\int\sqrt{x^2+\frac{1}{x^2}}\,d\left[(x+\frac{1}{x})+(x-\frac{1}{x})\right]$

$= \frac{1}{2}\int\sqrt{(x+\frac{1}{x})^2-2}\,d(x+\frac{1}{x}) + \frac{1}{2}\int\sqrt{(x-\frac{1}{x})^2+2}\,d(x-\frac{1}{x})$

$= \frac{1}{4}(x+\frac{1}{x})\sqrt{x^2+\frac{1}{x^2}} - \frac{1}{2}\ln\left|x+\frac{1}{x}+\sqrt{x^2+\frac{1}{x^2}}\right| + \frac{1}{4}(x-\frac{1}{x})\sqrt{x^2+\frac{1}{x^2}} + \frac{1}{2}\ln\left|x-\frac{1}{x}+\sqrt{x^2+\frac{1}{x^2}}\right| + C$

$= \frac{x}{2}\sqrt{x^2+\frac{1}{x^2}} + \frac{1}{2}\ln\left|\frac{x^2-1+\sqrt{x^4+1}}{x^2+1+\sqrt{x^4+1}}\right| + C$

(2) $\int\frac{x\,dx}{\sqrt{1+\sqrt[3]{x^2}}} = \frac{1}{2}\int\frac{dx^2}{\sqrt{1+\sqrt[3]{x^2}}}$ 令 $\sqrt{1+\sqrt[3]{x^2}} = t$ $\therefore x^2 = (t^2-1)^3$

$\therefore \int\frac{x\,dx}{\sqrt{1+\sqrt[3]{x^2}}} = \frac{1}{2}\int\frac{d(t^2-1)^3}{t} = \cancel{\frac{1}{2}}\int\frac{1}{\cancel{t}}\cdot 3(t^2-1)^2\cdot\cancel{2t}\,dt$

$= 3\int(t^4-2t^2+1)\,dt$

$= 3\int t^4dt - 6\int t^2dt + \int 3dt$

$= \frac{3}{5}t^5 - 2t^3 + 3t + C$

$= \frac{3}{5}(1+\sqrt[3]{x^2})^{\frac{5}{2}} - 2(1+\sqrt[3]{x^2})^{\frac{3}{2}} + 3\sqrt{1+\sqrt[3]{x^2}} + C$

8.

(1) 解：令 $I=\int\frac{dx}{1+x^4}$ $J=\int\frac{x^2dx}{1+x^4}$

$$\therefore I+J=\int\frac{x^2+1}{x^4+1}dx=\int\frac{x^2+1}{(x^2+1)^2-(\sqrt{2}x)^2}dx=\frac{1}{2}\int\frac{1}{x^2+1+\sqrt{2}x}+\frac{1}{2}\int\frac{1}{x^2+1-\sqrt{2}x}$$

$$=\frac{1}{2}\int\frac{dx}{(x+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}+\frac{1}{2}\int\frac{dx}{(x-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}=\frac{\sqrt{2}}{2}\int\frac{d(\sqrt{2}x+1)}{1+(\sqrt{2}x+1)^2}+\frac{\sqrt{2}}{2}\int\frac{d(\sqrt{2}x-1)}{1+(\sqrt{2}x-1)^2}$$

$$=\frac{\sqrt{2}}{2}\left[\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right]+C_1$$

$$J-I=\int\frac{x^2-1}{x^4+1}dx=\int\frac{1-\frac{1}{x^2}}{(x+\frac{1}{x})^2-2}dx=\int\frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}\xlongequal{\text{令}t=x+\frac{1}{x}}\int\frac{dt}{t^2-2}=\frac{1}{2\sqrt{2}}\int\left(\frac{1}{t-\sqrt{2}}-\frac{1}{t+\sqrt{2}}\right)dt$$

$$=\frac{\sqrt{2}}{4}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C_2=\frac{\sqrt{2}}{4}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+C_2$$

$$\therefore\begin{cases}I+J=\frac{\sqrt{2}}{2}\left[\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right]+C_1\\ J-I=\frac{\sqrt{2}}{4}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right)+C_2\end{cases}$$

$$\therefore I=\frac{\sqrt{2}}{4}\left[\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right]-\frac{\sqrt{2}}{8}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C$$

$$J=\frac{\sqrt{2}}{4}\left[\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right]+\frac{\sqrt{2}}{8}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+C$$

8. (2)

令 $I=\int\frac{\sin x\,dx}{a\cos x+b\sin x}$，$J=\int\frac{\cos x\,dx}{a\cos x+b\sin x}$

$\therefore bI+aJ=\int 1\,dx=x+C_1$

而 $bI-aJ=\int\frac{b\sin x-a\cos x}{b\sin x+a\cos x}dx=\int\frac{\sqrt{a^2+b^2}\sin(x-\varphi)}{\sqrt{a^2+b^2}\sin(x+\varphi)}dx$

$=\int\frac{\sin(x+\varphi-2\varphi)}{\sin(x+\varphi)}dx=\int\cos2\varphi\,dx-\sin2\varphi\cdot\int\cot(x+\varphi)dx$

$=x\cos2\varphi-\sin2\varphi\cdot\ln|\sin(x+\varphi)|+C_2$ $\left(\tan\varphi=\frac{a}{b}\right)$

$\therefore I=\frac{x\cos^2\varphi}{b}-\frac{\sin\varphi\cos\varphi}{b}\ln|\sin(x+\varphi)|+C$

$J=\frac{x\sin^2\varphi}{a}+\frac{\sin\varphi\cos\varphi}{a}\ln|\sin(x+\varphi)|+C$ (其中 $\varphi=\arctan\frac{a}{b}$)

9.

(1)

解: $\int \frac{4x^5-1}{(x^5+x+1)^2}dx = \int \frac{4x^3-\frac{1}{x^2}}{(x^4+1+\frac{1}{x})^2}dx = \int \frac{d(x^4+\frac{1}{x}+1)}{(x^4+\frac{1}{x}+1)^2} = -\frac{1}{x^4+\frac{1}{x}+1}+C = \frac{-x}{x^5+x+1}+C$

(2)

解: $\int\sqrt{\frac{x}{1-x\sqrt{x}}}\,dx = \int\frac{\sqrt{x}\,dx}{\sqrt{1-x\sqrt{x}}} = \frac{2}{3}\int\frac{d\,x\sqrt{x}}{\sqrt{1-x\sqrt{x}}} = -\frac{4}{3}\sqrt{1-x\sqrt{x}}+C$

(3) 令 $\ln(x+\sqrt{1+x^2})=t$ 则 $x+\sqrt{1+x^2}=e^t$

又 $\frac{1}{x+\sqrt{1+x^2}} = \sqrt{1+x^2}-x = e^{-t}$ $\therefore \begin{cases} x = \frac{e^t-e^{-t}}{2} \\ \sqrt{1+x^2} = \frac{e^t+e^{-t}}{2} \end{cases}$

$$\therefore \int \frac{\ln(x+\sqrt{1+x^2})}{(1+x^2)^{\frac{3}{2}}}dx = \int \frac{8t}{(e^t+e^{-t})^3}\,d\frac{e^t-e^{-t}}{2} = \int\frac{4t\,dt}{(e^t+e^{-t})^2} = \int\frac{4te^{2t}dt}{(e^{2t}+1)^2}$$

$$= \int\frac{2t\,de^{2t}}{(e^{2t}+1)^2} = -\int 2t\,d\frac{1}{e^{2t}+1} = -\left(\frac{2t}{e^{2t}+1} - \int\frac{d2t}{e^{2t}+1}\right)$$

$$= \frac{-2t}{e^{2t}+1} + \int\frac{1+e^{2t}-e^{2t}}{e^{2t}+1}d2t = \frac{-2t}{e^{2t}+1} + \int 1\,d2t - \int\frac{de^{2t}}{e^{2t}+1}$$

$$= \frac{-2t}{e^{2t}+1} + 2t - \ln|e^{2t}+1| + C$$

$$= \frac{-2\ln(x+\sqrt{1+x^2})}{2(x^2+x\sqrt{1+x^2}+1)} + 2\ln(x+\sqrt{1+x^2}) - \ln(2x^2+2x+2) + C$$

$$= \frac{(x^2+x\sqrt{1+x^2})\ln(x+\sqrt{1+x^2})}{x^2+x\sqrt{1+x^2}+1} + \ln\frac{x+\sqrt{1+x^2}}{2x^2+2x+2} + C$$

(4)
$$\int \frac{\sqrt{e^x-1}}{\sqrt{e^x+1}}dx=\int\frac{e^x-1}{\sqrt{e^{2x}-1}}dx=\int\frac{de^x}{\sqrt{e^{2x}-1}}-\int\frac{dx}{\sqrt{e^{2x}-1}}$$
$$=\ln\left|e^x+\sqrt{e^{2x}-1}\right|-\int\frac{1}{e^{2x}}\cdot\frac{2e^{2x}}{2\sqrt{e^{2x}-1}}dx$$
$$=\ln\left|e^x+\sqrt{e^{2x}-1}\right|-\int\frac{1}{e^{2x}}d\sqrt{e^{2x}-1}=\ln\left|e^x+\sqrt{e^{2x}-1}\right|-\int\frac{d\sqrt{e^{2x}-1}}{1+(\sqrt{e^{2x}-1})^2}$$
$$=\ln\left|e^x+\sqrt{e^{2x}-1}\right|-\arctan\sqrt{e^{2x}-1}+C$$

(5)
$$\int\frac{dx}{\sqrt{1+e^x}+\sqrt{1-e^x}}=\int\frac{\sqrt{1+e^x}-\sqrt{1-e^x}}{2e^x}dx=\int\frac{\sqrt{1+e^x}}{2e^x}dx-\int\frac{\sqrt{1-e^x}}{2e^x}dx.$$

令 $\sqrt{1+e^x}=t$ $\therefore e^x=t^2-1$, $x=\ln(t^2-1)$

$$\therefore\int\frac{\sqrt{1+e^x}}{2e^x}dx=\int\frac{t}{2(t^2-1)}\cdot\frac{2t}{t^2-1}dt=\int\frac{t^2}{(t^2-1)^2}dt=\int\frac{t^2-1+1}{(t^2-1)^2}dt$$
$$=\int\frac{dt}{t^2-1}+\int\frac{dt}{(t^2-1)^2}=\frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt+\frac{1}{4}\int\left(\frac{1}{t+1}-\frac{1}{t-1}+\frac{1}{(t+1)^2}+\frac{1}{(t-1)^2}\right)dt$$
$$=\frac{1}{4}\int\frac{dt}{t-1}-\frac{1}{4}\int\frac{dt}{t+1}+\frac{1}{4}\int\frac{dt}{(t+1)^2}+\frac{1}{4}\int\frac{dt}{(t-1)^2}$$
$$=\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|-\frac{t}{2(t^2-1)}+C=\frac{1}{4}\ln\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}-\frac{\sqrt{e^x+1}}{2e^x}+C$$

同理可得 $\int\frac{\sqrt{1-e^x}}{2e^x}dx=\frac{1}{4}\ln\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1}\right|+\frac{\sqrt{1-e^x}}{2e^x}+C$

$$\therefore\int\frac{dx}{\sqrt{1+e^x}+\sqrt{1-e^x}}=\frac{1}{4}\ln\left|\frac{(\sqrt{1+e^x}-1)(\sqrt{1-e^x}-1)}{(\sqrt{1+e^x}+1)(\sqrt{1-e^x}-1)}\right|-\frac{1}{2e^x}(\sqrt{1+e^x}-\sqrt{1-e^x})+C$$

(6)

$$\int\frac{1+\sin x}{1+\cos x}e^x dx=\int\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}e^x dx=\int\frac{e^x d\frac{x}{2}}{\cos^2\frac{x}{2}}+\int\tan\frac{x}{2}e^x dx$$

$$=\int e^x d\tan\frac{x}{2}+\int\tan\frac{x}{2}e^x dx=e^x\tan\frac{x}{2}-\int\tan\frac{x}{2}e^x dx+\int\tan\frac{x}{2}e^x dx$$

$$=e^x\tan\frac{x}{2}+C$$

(7)

$$\int\frac{1+x}{x(1+xe^x)}dx=\int\frac{1+xe^x+x-xe^x}{x(1+xe^x)}=\int\frac{dx}{x}+\int\frac{1-e^x}{1+xe^x}dx=\int\frac{dx}{x}-\int\frac{1-e^{-x}}{x+e^{-x}}dx$$

$$=\int\frac{dx}{x}-\int\frac{d(x+e^{-x})}{x+e^{-x}}=\ln|x|-\ln|x+e^{-x}|+C=\ln\left|\frac{x}{x+e^{-x}}\right|+C$$

(8)

$$\int\frac{x+\sin x}{1+\cos x}dx=\int\frac{x\,dx}{1+\cos x}+\int\frac{\sin x\,dx}{1+\cos x}$$

其中 $\int\frac{x\,dx}{1+\cos x}=\int\frac{x\,dx}{2\cos^2\frac{x}{2}}=\int\frac{x\,d\frac{x}{2}}{\cos^2\frac{x}{2}}=\int x\,d\tan\frac{x}{2}=x\tan\frac{x}{2}-\int\tan\frac{x}{2}dx$

$$=x\tan\frac{x}{2}-2\int\tan\frac{x}{2}d\frac{x}{2}=x\tan\frac{x}{2}-2\ln\left|\tan\frac{x}{2}\right|+C.$$

$$\int\frac{\sin x\,dx}{1+\cos x}=-\int\frac{d(\cos x+1)}{\cos x+1}=-\ln(1+\cos x)+C$$

$$\therefore\int\frac{x+\sin x}{1+\cos x}dx=x\tan\frac{x}{2}-\ln\frac{2\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+C.$$

(9)

$$\int\frac{x\sin x}{\cos^3 x}dx=-\int\frac{x\,d\cos x}{\cos^3 x}=\frac{1}{2}\int x\,d\frac{1}{\cos^2 x}=\frac{1}{2}\left(\frac{x}{\cos^2 x}-\int\frac{dx}{\cos^2 x}\right)$$

$$=\frac{x}{2\cos^2 x}-\frac{1}{2}\tan x+C$$

(10)

令 $\arcsin\frac{x}{x+1}=t$ ∴ $\sin t=\frac{x}{x+1}$，∴ $x=\frac{\sin t}{1-\sin t}$

∴ $\int \arcsin\frac{x}{x+1}dx=\int t\cdot\frac{\cos t\,dt}{(1-\sin t)^2}=\int\frac{t\,d(\sin t-1)}{(\sin t-1)^2}$

$=-\int t\,d\frac{1}{\sin t-1}=-\left(\frac{t}{\sin t-1}-\int\frac{dt}{\sin t-1}\right)$

$=-\frac{t}{\sin t-1}-\int\frac{1+\sin t}{1-\sin^2 t}dt$

$=-\frac{t}{\sin t-1}-\int\frac{dt}{\cos^2 t}-\int\tan t\sec t\,dt=-\frac{t}{\sin t-1}-\tan t+\int\frac{d\cos t}{\cos^2 t}$

$=\frac{-t}{\sin t-1}-\tan t-\frac{1}{\cos t}+C$

$=(x+1)\arcsin\frac{x}{x+1}-\tan\left[\arcsin\frac{x}{x+1}\right]-\frac{1}{\cos\left[\arcsin\frac{x}{x+1}\right]}+C$

(11) $\int\frac{dx}{\sinh x+2\cosh x}=\int\frac{dx}{\frac{e^x-e^{-x}}{2}+e^x+e^{-x}}=\int\frac{2de^x}{1+3e^{2x}}=\frac{2}{\sqrt3}\int\frac{d(\sqrt3e^x)}{1+(\sqrt3e^x)^2}$

$=\frac{2}{3}\sqrt3\arctan(\sqrt3e^x)+C$

(12) $\int\sinh x\sinh 2x\sinh 3x\,dx=\int\sinh x\cdot2\sinh x\cosh x\,(3\sinh x+4\sinh^3x)dx$

$=\int(6\sinh^3x+8\sinh^5x)\,d\sinh x=\frac{3}{2}\sinh^4x+\frac{4}{3}\sinh^6x+C$

(13) $\int\left[\frac{f}{f'}-\frac{f^2f''}{(f')^3}\right]dx=\int\frac{f}{f'}dx-\int\frac{f^2df'}{(f')^3}=\int\frac{f}{f'}dx+\frac{1}{2}\int f^2\,d\frac{1}{(f')^2}$

$=\int\frac{f}{f'}dx+\frac{1}{2}\left(\frac{f^2}{(f')^2}-\int\frac{2f\cdot f'}{(f')^2}dx\right)=\int\frac{f}{f'}dx+\frac{f^2}{2(f')^2}-\int\frac{f}{f'}dx$

$=\frac{f^2(x)}{2[f'(x)]^2}+C$