第八章 定积分

- 定积分定义

- 可积必有界.

- 常见可积函数类：单调有界/连续/存在有限间断点的有界函数

- 达布上和/达布下和 $S(T)$、$s(T)$ T为分割

  - 可积充要条件一：$\|T\| \to 0$时 达布上下和相等.
  - 可积充要条件二：对$\forall \varepsilon, \eta > 0$. 存在分割T使所有振幅$\geq \varepsilon$的小区间长度和$< \eta$.

积分第一中值定理　$f \in R[a,b]$且不变号　g在$[a,b]$连续.　则 $\int_a^b f(x)g(x)dx = g(\xi)\int_a^b f(x)dx$　$\xi \in [a,b]$

积分第二中值定理　$f \in R[a,b]$　g在$[a,b]$单调.　则 $\int_a^b f(x)g(x)dx = g(a)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x)dx$　$\xi \in [a,b]$

微积分基本定理

$$\left.\begin{array}{l} F(x) \triangleq \int_a^x f(t)dt,\ x \in [a,b] \\ f(x) \in R[a,b] \end{array}\right\} \Rightarrow F(x) \in C[a,b]$$

$+$ f在x_0连续 $\Rightarrow$ $F(x)$在x_0可导且$F'(x_0) = f(x_0)$

N-L公式.　$f \in C[a,b]$.　$F'(x) = f(x)$, $x \in [a,b]$.　$\Rightarrow \int_a^b f(t)dt = F(b) - F(a)$

练习8.1.

证：不妨证明只有一个点处与$f(x)$不同的情况，之后容易将其类比为有限个点。

设$f(x)\in R[a,b]$. $\exists x_0\in[a,b]$, $f(x_0)\neq g(x_0)$. 且$x\neq x_0$时 $f(x)=g(x)$.

只需证明：对$\forall\varepsilon>0$. $\exists\delta>0$. 对$\forall(T,\xi)$分割，只要$\Delta(T)<\delta$.

就有$\left|\sum g(\xi_i)\Delta x_i-\int_a^b f(x)dx\right|<\varepsilon$.

[证]: $\left|\sum g(\xi_i)\Delta x_i-\int_a^b f(x)dx\right|$

$\leq\left|\sum g(\xi_i)\Delta x_i-\sum f(\xi_i)\Delta x_i\right|+\left|\sum f(\xi_i)\Delta x_i-\int_a^b f(x)dx\right|$

$\leq|g(x_0)-f(x_0)|\Delta T+\varepsilon$.

取$\delta=\dfrac{\varepsilon}{|g(x_0)-f(x_0)|}$ $\because\delta>\Delta T$

$\therefore$上式$<\varepsilon+\varepsilon=2\varepsilon$. 证毕.

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练习8.2.

1. 对于分割T. ∵对 $\forall x\in[x_{k-1},x_k)$ $(k=0,1,\cdots,n)$，$f(x)$ 恒为常数

∴ $\omega_k=0$ 对 $\forall\varepsilon>0$，

此时 $\sum_{k=1}^{n}\omega_k\Delta x_k=0<\varepsilon$ 由定理5，$f(x)\in R[a,b]$.

2. [必要性]：若f(x)在[a,b]上可积，则有 $\lim_{\Delta T\to 0}S(T)=\lim_{\Delta T\to 0}s(T)$.

构造函数：$C_2(x)=M_k$，$x\in[x_{k-1},x_k)$

$C_1(x)=m_k$，$x\in[x_{k-1},x_k)$.

满足条件1：$C_2(x)\le f(x)\le C_1(x)$

满足条件2：$\int_a^b C_1(x)dx=s(T)$，$\int_a^b C_2(x)dx=S(T)$

∵有 $\lim_{\Delta T\to 0}S(T)=\lim_{\Delta T\to 0}s(T)$.

∴对 $\forall\Delta T$：$\Delta(T)<\delta$，对于 $\forall\varepsilon>0$. 有 $0\le\int_a^b C_2(x)dx-\int_a^b C_1(x)dx<\varepsilon$

[充分性] 由1然，$C_1(x)$、$C_2(x)$ 均可积，对 $\forall\varepsilon>0$，$\exists\delta>0$

$\forall T$：$\Delta(T)<\delta$，有 $\left|\sum C_1(\xi_k)\Delta x_k-\int_a^b C_1(x)dx\right|<\varepsilon$，$\left|\sum C_2(\xi_k)\Delta x_k-\int_a^b C_2(x)dx\right|<\varepsilon$

$$\begin{aligned}\therefore \sum\omega_k\Delta x_k &\le \sum[C_2(\xi_k)-C_1(\eta_k)]\Delta x_k\\ &= \sum C_2(\xi_k)\Delta x_k-\sum C_1(\eta_k)\Delta x_k\\ &< \left(\int_a^b C_2(x)dx+\varepsilon\right)-\left(\int_a^b C_1(x)dx-\varepsilon\right)\\ &< 3\varepsilon\end{aligned}$$

∴ $f(x)\in R[a,b]$

练习8.2.

3. 证:

对于给定的 $\varepsilon, \eta > 0$. 取 n 充分大. 使 $\delta = \frac{1}{n+\frac{1}{2}} < \frac{\varepsilon}{2}$. 显然 $\frac{1}{n+1} < \delta < \frac{1}{n}$.

对于 $f(x)$, 其在 $[0,1]$ 上的间断点是所有的 $\frac{1}{k}$ $(k \in N^*)$

将 $[0,1]$ 分为 $[0,\delta]$ 与 $[\delta,1]$

对于 $[\delta, 1]$:

其上的间断点集合 $\{\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}\}$.

对 $\forall \frac{1}{i}$ $(i=2,3,\cdots,n)$ 作开区间 $S_i = (\frac{1}{i} - \Delta\varepsilon_i, \frac{1}{i} + \Delta\varepsilon_i)$

所作的 $\Delta\varepsilon_i > 0$, 其足够小, 需满足: $\begin{cases} \text{任意 } S_i \text{ 与 } S_j \text{ 不相交 } (i \neq j) \\ \sum_{i=2}^{n} \Delta\varepsilon_i < \frac{\varepsilon}{2} \end{cases}$

记 $S_1 = [0, \delta)$. 记 $S = \bigcup_{i=1}^{n} S_i$

$\therefore [0,1] \setminus S = [\delta, \frac{1}{n} - \Delta\varepsilon_n] \cup [\frac{1}{n} + \Delta\varepsilon_n, \frac{1}{n-1} - \Delta\varepsilon_{n-1}] \cup \cdots \cup [\frac{1}{2} + \Delta\varepsilon_2, 1]$

记 $I_k = [\frac{1}{k+1} + \Delta\varepsilon_{k+1}, \frac{1}{k} - \Delta\varepsilon_k]$, $(k=2,\cdots,n-1)$, $I_1 = [\frac{1}{2} + \Delta\varepsilon_2, 1]$, $I_k = [\delta, \frac{1}{n} - \Delta\varepsilon_n]$

对 $\forall I_k$, $f(x)$ 在其上连续. 故一致连续

故 $\forall I_k$ 可被分为一系列区间 $\{D_{km}\}_{m=1,2,\cdots,h_k}$

在 $\forall D_{km}$ 上. $f(x)$ 振幅 $< \eta$.

取以上 $\{D_{km}\}_{m=1,2,\cdots,h_k}^{k=1,2,\cdots n}$ 的所有端点 ξ_s 记 $\{\xi_s\}$ 为 $f(x)$ 在 $[0,1]$ 上一分割.

$[0,1]$ 分割而成的所有子区间 $\begin{cases} [0,1] \setminus S: \text{ 任一子区间振幅 } < \eta \\ S = S_1 \cup (\bigcup_{i=2}^{n-1} S_i): \text{ 其区间长度} = \delta + \sum_{i=2}^{n} \Delta\varepsilon_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{cases}$

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练习8.3.

1. 要证原命题，只需证 $\int_{-a}^{b}(x^2-ab)f(x)dx\le 0$.

即证 $\int_{-a}^{b}(x+a)(x-b)f(x)dx+(b-a)\int_{-a}^{b}xf(x)dx\le 0$

$\because \int_{-a}^{b}xf(x)=0$ 故只需证 $\int_{-a}^{b}(x+a)(x-b)f(x)dx\le 0$.

$\because y=(x+a)(x-b)$ 在 $[-a,b]$ 上连续. $f(x)$ 在 $[-a,b]$ 上非负可积

$\therefore \int_{-a}^{b}(x+a)(x-b)f(x)dx=(\xi+a)(\xi-b)\int_{-a}^{b}f(x)dx$

$\because \xi\in[-a,b]$ $\therefore (\xi+a)(\xi-b)\le 0$

$\because f(x)$ 在 $[a,b]$ 上非负 $\therefore \int_{-a}^{b}f(x)dx\ge 0$

$\therefore$ 原式 ≤ 0.

2. 证明:

$\because \int_{a}^{b}(x-\frac{a+b}{2})f(x)dx$

$=\int_{a}^{\frac{a+b}{2}}(x-\frac{a+b}{2})f(x)dx+\int_{\frac{a+b}{2}}^{b}(x-\frac{a+b}{2})f(x)dx$

$=f(\xi_1)\int_{a}^{\frac{a+b}{2}}(x-\frac{a+b}{2})dx+f(\xi_2)\int_{\frac{a+b}{2}}^{b}(x-\frac{a+b}{2})dx$

$=[f(\xi_2)-f(\xi_1)]\frac{(b-a)^2}{2}$

$\because a<\xi_1<\frac{a+b}{2}<\xi_2<b$. $f(x)$ 在 $[a,b]$ 递增 $\therefore f(\xi_1)<f(\xi_2)$

$\therefore$ 上式 ≥ 0.

03. (1). 记 $f(x)=\frac{x^n}{1+x}dx$. 易知 $f(x)$ 在 $[0,1]$ 递增

对 $\forall \varepsilon>0$

$$\int_0^1 f(x)dx=\int_0^{1-\varepsilon} f(x)dx+\int_{1-\varepsilon}^1 f(x)dx$$

$$=(1-\varepsilon)f(\xi_1)+\varepsilon f(\xi_2)$$

$$\le (1-\varepsilon)f(1-\varepsilon)+\varepsilon f(1)$$

$$=\frac{(1-\varepsilon)^{n+1}}{2-\varepsilon}+\frac{\varepsilon}{2}<(1-\varepsilon)^n+\frac{\varepsilon}{2}$$

$\because \lim\limits_{n\to\infty}(1-\varepsilon)^n=0$ $\therefore \exists N$, 当 $n>N$ 时. $(1-\varepsilon)^n<\frac{\varepsilon}{2}$.

此时. $\int_0^1 f(x)dx<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. $\therefore \lim\limits_{n\to\infty}\int_0^1 f(x)dx=0$

(2). $\because \sin x$ 在 $[n,n+p]$ 上连续 $\frac{1}{x}$ 在 $[n,n+p]$ 上可积 且不变号

$$\therefore \int_n^{n+p}\frac{\sin x}{x}dx=\sin\xi\cdot\int_n^{n+p}\frac{1}{x}dx$$

$$\therefore \left|\int_n^{n+p}\frac{\sin x}{x}dx\right|\le\int_n^{n+p}\frac{dx}{x}$$

$$\therefore \lim_{n\to\infty}\left|\int_n^{n+p}\frac{\sin x}{x}dx\right|\le\lim_{n\to\infty}\int_n^{n+p}\frac{1}{x}dx$$

$$=\lim_{n\to\infty}\ln|x|\Big|_n^{n+p}=\lim_{n\to\infty}\ln\left(1+\frac{p}{n}\right)=0$$

$$\therefore \lim_{n\to\infty}\int_n^{n+p}\frac{\sin x}{x}dx=0$$

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练习8.3.

4. (1)

解: $\exists \xi \in (a,b)$ s.t. $\int_a^b e^{-nx^2}dx = e^{-n\xi^2}(b-a)$

$\therefore \lim_{n\to\infty}\int_a^b e^{-nx^2} = \lim_{n\to\infty} e^{-n\xi^2}(b-a)$

$\because \lim_{n\to\infty} e^{-n\xi^2} \leq \lim_{n\to\infty} e^{-na^2} = 0$ $\therefore \lim_{n\to\infty} e^{-n\xi^2}(b-a) = 0$

(2)

解: $\exists \xi \in (n^2, n^2+n)$ s.t. $\int_{n^2}^{n^2+n} \frac{e^{-\frac{1}{x}}dx}{\sqrt{x}} = e^{-\frac{1}{\xi}}\int_{n^2}^{n^2+n}\frac{dx}{\sqrt{x}}$

$= e^{-\frac{1}{\xi}} \cdot 2\sqrt{x}\Big|_{n^2}^{n^2+n} = e^{-\frac{1}{\xi}} \cdot (2\sqrt{n^2+n} - n)$

$= e^{-\frac{1}{\xi}} \cdot 2\sqrt{n}(\sqrt{n+1} - \sqrt{n})$

$= e^{-\frac{1}{\xi}} \cdot \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = e^{-\frac{1}{\xi}} \cdot \frac{2}{\sqrt{1+\frac{1}{n}}+1}$

$\therefore \lim_{n\to\infty}\int_{n^2}^{n^2+n}\frac{e^{-\frac{1}{x}}}{\sqrt{x}}dx = \lim_{n\to\infty} e^{-\frac{1}{\xi}} \frac{2}{\sqrt{1+\frac{1}{n}}+1}$

$= \lim_{n\to\infty} e^{-\frac{1}{\xi}}$ $\because e^{-\frac{1}{n^2}} \leq e^{-\frac{1}{\xi}} \leq e^{-\frac{1}{n^2+n}}$

且 $\lim_{n\to\infty} e^{-\frac{1}{n^2}} = \lim_{n\to\infty} e^{-\frac{1}{n^2+n}} = 1$ $\therefore \lim_{n\to\infty} e^{-\frac{1}{\xi}} = 1$

练习8.4.

1.

证明：$\because f(x) \in R[a,b]$

$\therefore$ 令 $g(x) = \int_a^x f(t)dt \quad \in C[a,b]$

$g(a) = \int_a^a f(t)dt = 0. \quad g(b) = \int_a^b f(t)dt$

$\because \quad \frac{1}{2}\left|\int_a^b f(t)dt\right| < \left|\int_a^b f(t)dt\right|$ 且 $\int_a^b f(x)dx \neq 0$

$\therefore \quad \frac{1}{2}\int_a^b f(x)dx$ 介于 $g(a)$ 与 $g(b)$ 值之间

由连续函数闭区间上的介值定理.

$\exists \xi \in (a,b) \quad s.t. \quad g(\xi) = \frac{1}{2}g(b)$

即 $\int_a^\xi f(x)dx = \int_\xi^b f(x)dx$

若去掉条件 $\int_a^b f(x)dx \neq 0$，显然结论不成立. 反例如：

取 $f(x) = x$，$a=-1, b=1$. 在 $(-1,1)$ 上不存在 ξ 使 $\int_a^\xi f(x)dx = \frac{1}{2}\int_a^b f(x)dx = 0$

2. (1) $\frac{dy}{dx}=\frac{dx^2}{dx}\cdot\frac{dy}{dx}=2x\cdot\sqrt{1+x^4}$

(2) $\frac{dy}{dx}=\left(\int_0^{x^4}\frac{dt}{\sqrt{1+t^4}}-\int_0^{x^2}\frac{dt}{\sqrt{1+t^4}}\right)=\frac{4x^3}{\sqrt{1+x^{16}}}-\frac{2x}{\sqrt{1+x^8}}$

(3) $\frac{dy}{dx}=\left(\int_0^{\cos x}\cos(\pi t^2)dt-\int_0^{\sin x}\cos(\pi t^2)dt\right)$

$=-\sin x\cdot\cos(\pi\cos^2x)-\cos x\cdot\cos(\pi\sin^2x)=\cos(\pi\cos^2x)\cdot(\cos x-\sin x).$

3. (1) $\int_{-\pi}^{\pi}\sin^2mx\,dx=\left[\frac{1}{2}x-\frac{\sin(2mx)}{4m}\right]_{-\pi}^{\pi}=\pi$

(2) $\int_{-\pi}^{\pi}\cos^2mx\,dx=\left[\frac{1}{2}x+\frac{\sin(2mx)}{4m}\right]_{-\pi}^{\pi}=\pi$

(3) $\int_{-\pi}^{\pi}\cos mx\cos nx\,dx=\left[\frac{\sin(m+n)x}{2(m+n)}+\frac{\sin(m-n)x}{2(m-n)}\right]_{-\pi}^{\pi}=0$

(4) $\int_{-\pi}^{\pi}\sin mx\sin nx\,dx=\left[\frac{\sin(m-n)x}{2(m-n)}-\frac{\sin(m+n)x}{2(m+n)}\right]\Bigg|_{-\pi}^{\pi}=0$

(5) 由于 $\sin mx\cos nx$ 为奇函数，显然原式为0

4.

(1) $\int_0^2 |1-x|dx = \int_0^1 (1-x)dx + \int_1^2 (x-1)dx = (x-\frac{x^2}{2})\Big|_0^1 + (\frac{x^2}{2}-x)\Big|_1^2 = 1$

(2) $\int \frac{(x+1)(x^2-3)}{3x^2}dx = \int(\frac{x}{3}+\frac{1}{3}-\frac{1}{x}-\frac{1}{x^2})dx = \frac{x^2}{6}+\frac{x}{3}-\ln x+\frac{1}{x}+C$

$\therefore \int_1^2 \frac{(x+1)(x^2-3)}{3x^2}dx = \frac{2}{3}\times 2 - \ln 2 + \frac{1}{2} - \frac{1}{2} - 1 = \frac{1}{3} - \ln 2$

(3). $\int \frac{(x-1)^4}{(x+1)^4}dx = \int \frac{(x-2)^4}{x^4}dx = \int \frac{x^4-8x^3+24x^2-32x+16}{x^4}dx = (x+1) - 8\ln(x+1) - \frac{24}{x+1}+\frac{16}{(x+1)^2}-\frac{16}{3(x+1)^3}+C$

$\therefore \int_0^1 (\frac{x-1}{x+1})^4 dx = 2-8\ln 2-12+4-\frac{2}{3}-(1-24+16-\frac{16}{3}) = \frac{17}{3}-8\ln 2$

(4). $\int x^2\cos x\, dx = x^2\sin x - 2\int x\sin x\, dx = x^2\sin x+2x\cos x-2\sin x+C$

$\therefore \int_0^{2\pi} x^2\cos x\, dx = 4\pi$

(5) $\int x^2\sin^2 x\, dx = \int x^2(\frac{1-\cos 2x}{2})dx = \frac{1}{2}\int x^2dx - \frac{1}{2}\int x^2\cos 2x\, dx \overset{由(4)}{=} \frac{x^3}{6}-\frac{1}{16}(4x^2\sin 2x+4x\cos 2x-2\sin 2x)+C$

$\therefore \int_0^\pi x^2\sin^2 x\, dx = \frac{\pi^3}{6}-\frac{1}{16}\cdot 4\pi = \frac{\pi^3}{6}-\frac{\pi}{4}$

(6) $\int x^3\ln^3 x\, dx = \frac{1}{4}x^4\ln^3 x - \frac{3}{4}\int x^3\ln x\, dx = \frac{x^4}{4}\ln^3 x - \frac{3x^4}{16}\ln^2 x + \frac{3}{8}\int x^3\ln x\, dx = \frac{x^4}{4}\ln^3 x - \frac{3x^4}{16}\ln^2 x+\frac{3x^4}{32}\ln x - \frac{3x^4}{128}+C$

$\therefore \int_1^e x^3\ln^3 x\, dx = (\frac{1}{4}-\frac{3}{16}+\frac{3}{32}-\frac{3}{128})e^4+\frac{3}{128} = \frac{17e^4}{128}+\frac{3}{128}$

(7) $\int e^x\cos^2 x\,dx = e^x\cos^2 x + \int e^x \sin 2x\,dx$

其中 $\int e^x\sin 2x\,dx = e^x\sin 2x - 2\int e^x\cos 2x\,dx = e^x\sin 2x - 2e^x\cos 2x\,dx - 4\int e^x\sin 2x\,dx$

$\therefore \int e^x\sin 2x\,dx = \frac{1}{5}(e^x\sin 2x - 2e^x\cos 2x)$ $\quad\therefore \int e^x\cos^2 x\,dx = e^x\cos^2 x + \frac{1}{5}(e^x\sin 2x - 2e^x\cos 2x) + C$

$\therefore \int_0^\pi e^x\cos^2 x\,dx = e^\pi + \frac{1}{5}\cdot(-2e^\pi) - \left[1+\frac{1}{5}\times(-2)\right] = \frac{3}{5}e^\pi - \frac{3}{5}$

(8). $\int \arcsin\frac{x}{x+1}\,dx = \int \arcsin\frac{x}{x+1}\,d(x+1) = (x+1)\arcsin\frac{x}{x+1} - \int (x+1)\cdot\frac{\frac{1}{(x+1)^2}}{\sqrt{1-\left(\frac{x}{x+1}\right)^2}}\,dx = (x+1)\arcsin\frac{x}{x+1} - \int\frac{dx}{\sqrt{1+2x}}$

$= (x+1)\arcsin\frac{x}{x+1} - \sqrt{1+2x} + C$ $\quad\therefore \int_0^3 \arcsin\frac{x}{1+x}\,dx = 4\arcsin\frac{3}{4} - \sqrt{7} + 1$

(9). $\int\frac{dx}{x^2-2x\cos\alpha+1} = \frac{1}{\sin\alpha}\int\frac{d\,\frac{x-\cos\alpha}{\sin\alpha}}{1+\left(\frac{x-\cos\alpha}{\sin\alpha}\right)^2} = \frac{1}{\sin\alpha}\arctan\frac{x-\cos\alpha}{\sin\alpha} + C$

$\therefore \int_{-1}^1\frac{dx}{x^2-2x\cos\alpha+1} = \frac{1}{\sin\alpha}\left(\arctan\frac{1-\cos\alpha}{\sin\alpha} + \arctan\frac{1+\cos\alpha}{\sin\alpha}\right) = \frac{\pi}{2\sin\alpha}$ $\quad$ (由于 $\frac{1-\cos\alpha}{\sin\alpha}\cdot\frac{1+\cos\alpha}{\sin\alpha} = 1$)

(10). $I_1 = \pi$ $\quad I_2 = \int_0^\pi 2\cos x\,dx = 0$. $\quad n \geq 3$时:

$I_n = \int_0^\pi \frac{\sin(n-2)x\cdot\cos 2x + \cos(n-2)x\cdot\sin 2x}{\sin x}\,dx = \int_0^\pi \frac{\sin(n-2)x\cdot(1-2\sin^2 x) + 2\sin x\cos x\cdot\cos(n-2)x}{\sin x}\,dx$

$= I_{n-2} + 2\int_0^\pi\left(\cos x\cos(n-2)x - \sin x\sin(n-2)x\right)dx = I_{n-2} + 2\int_0^\pi \cos(n-1)x\,dx$

$= I_{n-2} + \frac{2}{n-1}\sin(n-1)x\Big|_0^\pi = I_{n-2}$

$\therefore I_n = \begin{cases} 0 & n为偶 \\ \pi & n为奇 \end{cases}$

5. (1) 原式 $= \lim\limits_{n\to\infty}\left(\dfrac{\frac{1}{n}}{1+x^2\cdot(\frac{1}{n})^2}+\dfrac{\frac{1}{n}}{1+x^2\cdot(\frac{2}{n})^2}+\dfrac{\frac{1}{n}}{1+x^2\cdot(\frac{3}{n})^2}+\cdots+\dfrac{\frac{1}{n}}{1+x^2\cdot(\frac{n}{n})^2}\right)$

$= \int_0^1 \dfrac{dt}{1+x^2t^2}$

$\because \int \dfrac{dt}{1+x^2t^2} = \dfrac{1}{x}\arctan(xt)+C$

$\therefore$ 原式 $= \dfrac{\arctan x}{x}$

(2) 原式 $= \lim\limits_{n\to\infty}\dfrac{1}{n}\left[(\frac{1}{n})^p+(\frac{2}{n})^p+\cdots+(\frac{n}{n})^p\right]$

$= \int_0^1 x^p\,dx$ $\qquad \because \int x^p dx = \dfrac{x^{p+1}}{p+1}+C$

$\therefore$ 原式 $= \dfrac{1}{p+1}$

(3) 原式 $= \lim\limits_{n\to\infty}\dfrac{1}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2}{n}\pi+\cdots+\sin\frac{n-1}{n}\pi+\sin\frac{n}{n}\pi\right]$

$= \int_0^1 \sin\pi x\,dx$ $\qquad \because \int \sin\pi x\,dx = -\dfrac{1}{\pi}\cos\pi x$

$\therefore$ 原式 $= \dfrac{2}{\pi}$

练习8.5

1. (1) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_{-\frac{\pi}{2}}^{0} f(\sin(x+\frac{\pi}{2}))\,d(x+\frac{\pi}{2}) = \int_{-\frac{\pi}{2}}^{0} f(\cos x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

(2) $\int_0^{\pi} x f(\sin x)dx = \int_{\pi}^{0} (\pi-x) f(\sin(\pi-x))\,d(\pi-x) = \int_0^{\pi} (\pi-x) f(\sin x)dx$

$= \pi\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} x f(\sin x)dx \qquad \therefore \int_0^{\pi} x f(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$

2. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\frac{\pi}{2}} \frac{x\sin x}{1+\cos^2 x}dx + \int_{\frac{\pi}{2}}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

令 $x=\pi-t$. $\therefore \int_{\frac{\pi}{2}}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\frac{\pi}{2}} \frac{(\pi-t)\sin t}{1+\cos^2 t}dt$

$\therefore$ 原式 $= \pi\int_0^{\frac{\pi}{2}} \frac{\sin t\,dt}{1+\cos^2 t} + \int_0^{\frac{\pi}{2}} \frac{x\sin x}{1+\cos^2 x}dx - \int_0^{\frac{\pi}{2}} \frac{x\sin x}{1+\cos^2 x}dx$

$= \pi\int_0^{\frac{\pi}{2}} \frac{\sin t\,dt}{1+\cos^2 t} = -\pi\int_0^{\frac{\pi}{2}} \frac{d\cos t}{1+\cos^2 t}$

$= -\pi \arctan(\cos t)\Big|_0^{\frac{\pi}{2}} = (\arctan 1)\pi$

南開大學 作业纸

系别________ 班级________ 姓名________ 第 页

3. (1) $x \to \ln(x+1)$. $\int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \int_0^1 \frac{\sqrt{x}}{x+1} dx = \int_0^1 \frac{2x^2}{1+x^2} dx = 2 - 2\arctan x \Big|_0^1 = 2 - \frac{\pi}{2}$

(2) $x \to a\sin x$. $\int_0^a x^2\sqrt{a^2-x^2}\, dx = \int_0^{\frac{\pi}{2}} a^4 \sin^2 x \cos^2 x\, dx = \frac{a^4}{8} \cdot (x - \frac{1}{4}\sin 4x) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{16} a^4$

(3) $x \to \sqrt{3}\tan x + 1$ $\int_1^2 \frac{dx}{(x^2-2x+4)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{d(\sqrt{3}\tan x)}{(3\tan^2 x + 3)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3} \cdot \frac{1}{\cos^2 x}}{3\sqrt{3} \cdot \frac{1}{\cos^3 x}} dx = \int_0^{\frac{\pi}{6}} \frac{\cos x}{3} dx = \frac{1}{6}$

(4) $x \to e^x$ $\int_e^{e^2} \frac{dx}{x(\ln x)^3} = \int_1^2 \frac{e^x dx}{e^x \cdot x^3} = \int_1^2 \frac{dx}{x^3} = -\frac{1}{2x^2} \Big|_1^2 = \frac{3}{8}$

(5) $x \to \frac{\pi}{2} - x$ $\int_0^{2\pi} \sqrt{1+\sin x}\, dx = \int_{\frac{\pi}{2}}^{-\frac{3}{2}\pi} \sqrt{1+\cos x}\, (-dx) = \int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} \sqrt{1+\cos x}\, dx = \int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{2} \left|\cos \frac{x}{2}\right| dx$

$= \sqrt{2} \left(\int_{-\pi}^{\frac{\pi}{2}} \cos \frac{x}{2} dx - \int_{-\frac{3}{2}\pi}^{-\pi} \cos \frac{x}{2} dx \right) = 4\sqrt{2}$

(6) $\int \frac{x^2+1}{x^4+1} dx = \int \frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+2} dx = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2} = \int \frac{dx}{2+x^2} = \int \frac{\frac{1}{\sqrt{2}} d(\frac{x}{\sqrt{2}})}{(\frac{x}{\sqrt{2}})^2+1} = \frac{1}{\sqrt{2}} \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} + C$

$\therefore \int_{-1}^1 \frac{x^2+1}{x^4+1} dx = 2\int_0^1 \frac{x^2+1}{x^4+1} dx = \sqrt{2} \cdot \arctan \frac{x-\frac{1}{x}}{\sqrt{2}} = \sqrt{2}\left(0+\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}\pi$

(7) 令 $t = \tan \frac{x}{2}$

$\int \frac{dx}{(2+\cos x)(3+\cos x)} = \int \left(\frac{2}{t^2+3} - \frac{1}{t^2+2}\right) dt = \frac{2}{\sqrt{3}} \arctan \frac{\tan \frac{x}{2}}{\sqrt{3}} - \frac{1}{\sqrt{2}} \arctan \frac{\tan \frac{x}{2}}{\sqrt{2}} + C.$

$\therefore \int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)} = \left(\frac{2}{3}\sqrt{3} - \frac{\sqrt{2}}{2}\right)\pi$

南開大學 作业纸

系别________ 班级________ 姓名________ 第 页

(8). $\tan x \to x$ $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{1+\tan^4 x} = \int_0^1 \frac{1}{(1+x^4)(1+x^2)} dx = \int_0^1 \left[\frac{1}{2(1+x^2)} - \frac{x^2-1}{2(1+x^4)}\right] dx$

$\displaystyle = \frac{1}{2}\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2-1}{x^4+1} dx.$

$\displaystyle \because \int \frac{x^2-1}{x^4+1} dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx = \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2} = -\frac{1}{2\sqrt{2}} \ln\left|\frac{x+\frac{1}{x}+\sqrt{2}}{x+\frac{1}{x}-\sqrt{2}}\right| = \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C$

$\displaystyle \therefore 原式 = \frac{1}{2}\arctan x\Big|_0^1 - \frac{1}{2} \cdot \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\Bigg|_0^1 = \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{4\sqrt{2}} \cdot \ln\frac{2-\sqrt{2}}{2+\sqrt{2}} = \frac{\pi}{8} - \frac{\sqrt{2}}{8}\ln\frac{2-\sqrt{2}}{2+\sqrt{2}}$

(9) $x \to x+\frac{\pi}{2}$ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \int_{-\frac{\pi}{2}}^0 \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$

$\displaystyle \because \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} = 2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ $\therefore 原式 = \frac{\pi}{4}$

(10). $\displaystyle\int_0^2 \frac{x}{e^x+e^{2-x}} dx = \int_0^1 \frac{x}{e^x+e^{2-x}} dx + \int_1^2 \frac{x}{e^x+e^{2-x}} dx \triangleq I_1 + I_2$

$\displaystyle I_2 = \int_1^2 \frac{x-2}{e^x+e^{2-x}} dx + \int_1^2 \frac{2}{e^x+e^{2-x}} dx = \int_1^0 \frac{t}{e^{2-t}+e^t} dt + \int_1^2 \frac{2}{e^x+e^{2-x}} dx = -I_1 + \int_1^2 \frac{2}{e^x+e^{2-x}} dx$

$\displaystyle \therefore I_1 + I_2 = \int_1^2 \frac{2}{e^x+e^{2-x}} dx = \int_1^2 \frac{2e^x dx}{e^{2x}+e^2} = 2\int_e^{e^2} \frac{de^x}{e^{2x}+e^2} = \frac{2}{e}\arctan(e^{x-1})\Big|_1^2 = \frac{2}{e}\arctan(e) - \frac{\pi}{2e}$

(11). $x \to -x$. $\displaystyle\int_{-1}^1 \frac{dx}{(e^x+1)(x^2+1)} = \int_{-1}^1 \frac{d(-x)}{(e^{-x}+1)(x^2+1)} = -\int_{-1}^1 \frac{e^x dx}{(e^x+1)(x^2+1)}$

$\displaystyle \therefore 原式 = \frac{1}{2}\int_{-1}^1 \frac{e^x+1}{(e^x+1)(x^2+1)} dx = \frac{1}{2}\int_{-1}^1 \frac{dx}{1+x^2} = \frac{1}{2}\arctan x\Big|_{-1}^1 = \frac{\pi}{4}$

(12) $x \to \cos x$. $\displaystyle I_n = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx$ (课本 P276) $\displaystyle \therefore I_n = \frac{(2n)!!}{(2n+1)!!}$

南開大學 作业纸

系别________ 班级________ 姓名____________ 第 页

8(A)

1. 证：$\because g(x)$在$[a,b]$可积，对$\forall \varepsilon, \eta>0$，$\exists \delta_1>0$，对$\forall T: \Delta(T)<\delta_1$

有 $\sum\limits_{w_i^g \geq \eta} \Delta x_i < \varepsilon$

$\because f(x)$在$[a,b]$连续 $\therefore f(x)$在$[a,b]$一致连续 $\therefore$对$\forall \varepsilon>0$，$\exists \delta_2>0$，

对$\forall x', x'' \in [a,b]$且 $|g(x')-g(x'')| = w_i^g < \delta_2$时，

$|f(g(x'))-f(g(x''))| = w_i^{f\circ g} < \varepsilon$

取$\delta = \min\{\delta_1, \delta_2\}$，取上述$\delta_1 = \delta_2 = \delta$，$\eta = \delta$．设$f(g(x))$在$[a,b]$上最大振幅为$\Omega$

$\therefore \sum w_i^{f\circ g} \Delta x_i = \sum\limits_{w_i^g \geq \delta} w_i^{f\circ g} \Delta x_i + \sum\limits_{w_i^g < \delta} w_i^{f\circ g} \Delta x_i \leq \Omega \sum\limits_{w_i^g \geq \delta} \Delta x_i + \varepsilon \sum\limits_{w_i^g < \delta} \Delta x_i$

$\leq \Omega\varepsilon + (b-a)\varepsilon = (\Omega + b - a)\varepsilon$．$\therefore f\circ g$在$[a,b]$上可积

2.

(1) 不一定，反例如 $f(x) = \begin{cases} 1 & x \in Q \\ -1 & x \in R\backslash Q \end{cases}$

(2) 记$g(x) = f^3(x)$．

$\therefore f(x) = \sqrt[3]{g(x)}$

$\because y = \sqrt[3]{x}$连续，$g(x)$在$[a,b]$可积．

由1中结论 $f(x) = \sqrt[3]{g(x)}$在$[a,b]$可积

8-(A)3.

$\because f(x)$ 与 $g(x)$ 在 $[a,b]$ 可积，$\therefore f(x)$、$g(x)$ 在 $[a,b]$ 上有界. 设 $f(x)\le A$，$g(x)\le B$. $(x\in[a,b])$

证：$\because f(x)$、$g(x)$ 在 $[a,b]$ 均可积.

$\therefore$ 对 $\forall\varepsilon>0$，$\exists\delta_1>0$，只要 $\Delta(T)<\delta_1$，就有 $\sum_k \omega_k^f\cdot\Delta x_k<\varepsilon$

对 $\forall\varepsilon>0$. $\exists\delta_2>0$. 只要 $\Delta(T)<\delta_2$，就有 $\sum_k \omega_k^g\cdot\Delta x_k<\varepsilon$

$\because f(x)g(x)$ 也在 $[a,b]$ 可积

$$\therefore \int_a^b f(x)g(x)dx=\lim_{\Delta(T)\to0}\sum_k f(\xi_k)g(\xi_k)\Delta x_k.$$

要证原命题，即证 $\lim_{\Delta(T)\to0}\sum_k f(\xi_k)\Delta x_k\cdot[g(\theta_k)-g(\xi_k)]=0$.

取 $\delta=\min\{\delta_1,\delta_2\}$ 当 $\Delta(T)<\delta$ 时. 对 $\forall\varepsilon>0$.

$$\sum_k f(\xi_k)\Delta x_k\cdot[g(\theta_k)-g(\xi_k)]\le\sum_k f(\xi_k)\Delta x_k\cdot\omega_k^g$$

$$\le A\sum_k\omega_k^g\Delta x_k<A\varepsilon$$

$$\therefore\lim_{\Delta(T)\to0}\sum_k f(\xi_k)\Delta x_k\cdot[g(\theta_k)-g(\xi_k)]=0$$

即 $\lim_{\Delta(T)\to0}\sum_k f(\xi_k)g(\theta_k)\Delta x_k=\int_a^b f(x)g(x)dx$

5.

证：∵ $f(x)$ 是下凸函数 ∴ 对 $\forall \lambda\in[0,1]$，有：

$$\frac{1}{2}\left[f(x_1+\lambda(x_2-x_1))+f(x_2-\lambda(x_2-x_1))\right]\geq f\left(\frac{x_1+x_2}{2}\right)$$

两边对 λ 从 0 到 1 积分，右式 $=\int_0^1 f\left(\frac{x_1+x_2}{2}\right)d\lambda=f\left(\frac{x_1+x_2}{2}\right)$

左式：

~~哥~~ 令 $x_1+\lambda(x_2-x_1)=u$，$x_2-\lambda(x_2-x_1)=t$

$$左=\frac{1}{2}\cdot\frac{1}{x_2-x_1}\int_{x_1}^{x_2}f(u)du+\frac{1}{2}\cdot\frac{1}{x_2-x_1}\int_{x_1}^{x_2}f(t)dt$$

$$=\frac{1}{x_2-x_1}\int_{x_1}^{x_2}f(x)dx\geqslant f\left(\frac{x_1+x_2}{2}\right)=右 \text{ 成立.}$$

另：由 f 凸性，

$$\frac{1}{x_2-x_1}\int_{x_1}^{x_2}f(t)dt=\int_0^1 f(\lambda x_2+(1-\lambda)x_1)\,d\lambda$$

$$\leq\int_0^1[\lambda f(x_2)+(1-\lambda)f(x_1)]\,d\lambda=\frac{f(x_1)+f(x_2)}{2}. \text{ 得证.}$$

6. 把区间 $[0,1]$ n 等分. 由 $y=\ln x$ 的凸性，$\ln\left(\sum_{i=1}^{n}\frac{1}{n}f(\frac{i}{n})\right)\leq\sum_{i=1}^{n}\frac{1}{n}\ln\left(f(\frac{i}{n})\right)$

两边对 n 求极限：$\lim_{n\to\infty}\ln\left(\sum_{i=1}^{n}\frac{1}{n}f(\frac{i}{n})\right)\leq\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\ln\left(f(\frac{i}{n})\right)$

由 φ 的连续性，lim 与 f 可换序

$$\therefore 左式=\ln\left(\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}f(\frac{i}{n})\right)=\ln\int_0^1 f(x)dx$$

$$右式=\int_0^1\ln(f(x))dx$$

得证.

7\. case1 若$\int_a^b f^2 dx$ 与$\int_a^b g^2 dx$中至少有一者不为零：

不妨设 $\int_a^b f^2(x)dx \neq 0$ ∵对$\forall \lambda \in R$，在$[a,b]$上 $(\lambda f(x)-g(x))^2 \geq 0$

∴对$\forall \lambda \in R$，$[a,b]$上 有$\int_a^b (\lambda f(x)-g(x))^2 dx \geq 0$

展开：$\lambda^2 \int_a^b f^2(x) dx - 2\lambda \int_a^b f(x) g(x) dx + \int_a^b g^2(x) dx \geq 0$

将其视作关于λ的不等式 ∵其对$\forall \lambda \in R$都成立 ∴上式中$\Delta \leq 0$

∴ $\left(\int_a^b f(x) g(x) dx\right)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x) dx$.

case 2 若$\int_a^b f^2 dx = \int_a^b g^2 dx = 0$.

则 $\left|\int_a^b f(x)g(x)dx\right| \leq \int_a^b |f(x)g(x)|dx \leq \int_a^b \frac{f^2(x)+g^2(x)}{2}dx = \frac{1}{2}\left(\int_a^b f^2(x)dx + \int_a^b g^2(x)dx\right) = 0$

8\. 对区间$[a,b]$作分割T：$a = x_0 < x_1 < \cdots < x_n = b$. $\xi_i \in (x_{i-1}, x_i)$ $(i=1,\cdots,n)$
$\Delta x_i = x_i - x_{i-1}$

$$\because \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}} + \left(\int_a^b |g(x)|^2 dx\right)^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^n |f(\xi_i)\cdot\sqrt{\Delta x_i}|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^n |g(\xi_i)\cdot\sqrt{\Delta x_i}|^2\right)^{\frac{1}{2}}$$

直接应用 P169. 5(A) - 20(4) Minkowski 不等式，

$$\text{上式} \geq \left(\sum_{i=1}^n (|f(\xi_i)|+|g(\xi_i)|)^2 \Delta x_i\right)^{\frac{1}{2}}$$

$$= \left(\int_a^b (|f(x)|+|g(x)|)^2 dx\right)^{\frac{1}{2}}.$$

9.

证：

$$\left[\int_a^b f(x)\cos kx\,dx\right]^2+\left[\int_a^b f(x)\sin kx\,dx\right]^2$$ 应用上题(7)结论

$$\le \int_a^b (\sqrt{f(x)})^2\cdot\int_a^b(\sqrt{f(x)}\cos kx)^2+\int_a^b(\sqrt{f(x)})^2\int_a^b(\sqrt{f(x)}\sin kx)^2$$

$$=\int_a^b f(x)\cos^2 kx\,dx+\int_a^b f(x)\sin^2 kx\,dx$$

$$=\int_a^b f(x)\,dx=1$$

10. $\int_0^{2\pi} f(x)\sin(nx)dx = \sum_{k=0}^{n-1}\left[\int_{\frac{2k\pi}{n}}^{\frac{(2k+1)\pi}{n}} f(x)\sin nx\,dx + \int_{\frac{(2k+1)\pi}{n}}^{\frac{(2k+2)\pi}{n}} f(x)\sin nx\,dx\right]$

$x \to \frac{t+2k\pi}{n}$: $\int_{\frac{2k\pi}{n}}^{\frac{(2k+1)\pi}{n}} f(x)\sin nx\,dx = \frac{1}{n}\int_0^{\pi} f\left(\frac{2k\pi+t}{n}\right)\sin t\,dt$

$x \to \frac{t+(2k+1)\pi}{n}$: $\int_{\frac{(2k+1)\pi}{n}}^{\frac{(2k+2)\pi}{n}} f(x)\sin nx\,dx = -\frac{1}{n}\int_0^{\pi} f\left(\frac{(2k+1)\pi+t}{n}\right)\sin t\,dt$

$\because$ $f(x)$在$[0,2\pi]$递减，$\sin t$在$[0,\pi]$非负

$\therefore \left(\int_{\frac{2k\pi}{n}}^{\frac{(2k+1)\pi}{n}} + \int_{\frac{(2k+1)\pi}{n}}^{\frac{(2k+2)\pi}{n}}\right) f(x)\sin nx\,dx = \frac{1}{n}\cdot\int_0^{\pi}\sin t\cdot\left[f\left(\frac{2k\pi+t}{n}\right) - f\left(\frac{(2k+1)\pi+t}{n}\right)\right]dt \geq 0$

$\therefore \int_0^{2\pi} f(x)\sin(nx)dx \geq 0$

11. 设$M = \max\limits_{[a,b]} f(x)$. $M=0$时显然成立. 不妨考虑$M>0$.

对$\forall\, \varepsilon \in (0,M)$, $\exists [\alpha,\beta] \subseteq [a,b]$ 使 $M-\varepsilon \leq f(x) \leq M$, $\forall x \in [\alpha,\beta]$

$\therefore \left[\int_a^b f^n(x)dx\right]^{\frac{1}{n}} \geq \left[\int_\alpha^\beta f^n(x)dx\right]^{\frac{1}{n}} \geq (M-\varepsilon)(\beta-\alpha)^{\frac{1}{n}}$. 此外 $\left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} \leq M(b-a)^{\frac{1}{n}}$

$\therefore M(b-a)^{\frac{1}{n}} \geq \left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} \geq (M-\varepsilon)(\beta-\alpha)^{\frac{1}{n}}$

利用$\lim\limits_{n\to\infty}\sqrt[n]{a}=1$ $(a>0)$. 对上式取$n\to\infty$得:

$M \geq \overline{\lim\limits_{n\to\infty}}\left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} \geq \underline{\lim\limits_{n\to\infty}}\left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} \geq M-\varepsilon$. 由$\varepsilon$的任意性:

$\overline{\lim\limits_{n\to\infty}}\left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} = \underline{\lim\limits_{n\to\infty}}\left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} = M$. $\therefore \lim\limits_{n\to\infty}\left(\int_a^b f^n(x)dx\right)^{\frac{1}{n}} = M$

南開大學 作业纸

系别________ 班级________ 姓名________ 第 页

12. $\lim\limits_{n\to\infty} n\int_0^1 x^n f(x)dx - f(1) = \lim\limits_{n\to\infty}\left[n\int_0^1 x^n f(x)dx - f(1)\int_0^1 x^n dx\right]$

$= \lim\limits_{n\to\infty}\int_0^1 nx^n(f(x)-f(1))dx$

下证 $\lim\limits_{n\to\infty}\int_0^1 nx^n(f(x)-f(1))dx = 0$

对 $\forall \varepsilon > 0$

原式 $= \int_0^{1-\varepsilon} nx^n(f(x)-f(1))dx + \int_{1-\varepsilon}^1 nx^n(f(x)-f(1))dx$

$= [f(\xi_1)-f(1)]\frac{n}{n+1}(1-\varepsilon)^n + \varepsilon\cdot n\cdot\xi_2^n[f(\xi_2)-f(1)]$ $\xi_1、\xi_2\in[0,1]$

$\because f$ 在$[0,1]$上连续 $\therefore f$在$[0,1]$上有界. 记$W^f_{[0,1]} = \Omega$

$\therefore [f(\xi_1)-f(1)]\frac{n}{n+1}(1-\varepsilon)^n \le \Omega(1-\varepsilon)^n\cdot\frac{n}{n+1} \le \Omega(1-\varepsilon)^n$

$\lim\limits_{n\to\infty}\Omega(1-\varepsilon)^n = 0$

若取 $\varepsilon = \frac{\eta}{n}$ (η为$\forall$正数)

则 $\varepsilon\cdot n\cdot\xi_2^n[f(\xi_2)-f(1)] = \eta\cdot\xi_2^n[f(\xi_2)-f(1)] < \Omega\cdot\eta$

$\lim\limits_{n\to\infty}\Omega\eta = 0$

$\therefore \lim\limits_{n\to\infty}\int_0^1 nx^n(f(x)-f(1))dx < \lim\limits_{n\to\infty}[\Omega(1-\varepsilon)^n + \Omega\eta] = 0$

优化以上证明,不难有 $\left|\lim\limits_{n\to\infty}\int_0^1 nx^n(f(x)-f(1))dx\right| < \lim\limits_{n\to\infty}\Omega[(1-\varepsilon)^n+\Omega] = 0$

$\therefore \lim\limits_{n\to\infty}\int_0^1 nx^n(f(x)-f(1))dx = 0$

即 $\lim\limits_{n\to\infty}\int_0^1 nx^n f(x) = f(1)$

南開大學 作业纸

系别＿＿＿＿ 班级＿＿＿＿ 姓名＿＿＿＿ 第 页

13. $\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n = n\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n\right)$

$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^n + \frac{1}{n}$ 对于函数 $f(x)=x^n$.

在区间 $[0,1]$ 上作 n 等分，其达布下和 $s(T) = \frac{1}{n}\cdot\left(\frac{0}{n}\right)^n + \frac{1}{n}\cdot\left(\frac{1}{n}\right)^n + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^n + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^n$

$= \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^n$

$\therefore \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^n = s(T)+\frac{1}{n} \le \int_0^1 x^n\,dx + \frac{1}{n} = \frac{1}{n+1}+\frac{1}{n}$

$\therefore \sum_{k=1}^{n}\left(\frac{k}{n}\right)^n \le 1+\frac{n}{n+1} < 2$ 右式得证.

由(5) $\because f(x)=x^n$ 下凸 $\therefore \int_k^{k+1} f(x)dx \le \frac{f(k)+f(k+1)}{2}$

即: $\frac{(k+1)^n+k^n}{2} \ge \frac{1}{n+1}\left[(k+1)^{n+1}-k^{n+1}\right]$

分别令 $k=1,2,\cdots n-1$. 把 $n-1$ 个不等式相加，有:

左式 $= \frac{1}{2}+2^n+3^n+\cdots+(n-1)^n+\frac{n^n}{2} \ge$ 右式 $= \frac{1}{n+1}\left(2^{n+1}-1+3^{n+1}-2^{n+1}+\cdots+n^{n+1}-(n-1)^{n+1}\right]$

$= \frac{1}{n+1}\left(n^{n+1}-1\right)$

$\therefore 1^n+2^n+3^n+\cdots+n^n \ge \frac{1}{2}+\frac{n^n}{2}+\frac{n^{n+1}}{n+1}-\frac{1}{n+1}$

$\therefore \left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\cdots+\left(\frac{n}{n}\right)^n \ge \left(\frac{1}{2}-\frac{1}{n+1}\right)\cdot\frac{1}{n^n}+\frac{3n+1}{2n+2}$ 左式得证.

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系别＿＿＿＿ 班级＿＿＿＿ 姓名＿＿＿＿ 第 页

习题8(A).

15. $\lim\limits_{h\to0^+}\frac{1}{h}\int_a^x[f(t+h)-f(t)]dt$

$=\lim\limits_{h\to0^+}\frac{1}{h}\int_a^x f(t+h)dt-\lim\limits_{h\to0^+}\frac{\int_a^x f(t)}{h}dt$

$=\lim\limits_{h\to0^+}\frac{1}{h}\int_{a+h}^{x+h}f(t+h)d(t+h)-\lim\limits_{h\to0^+}\frac{1}{h}\int_a^x f(t+h)d(t+h)$

$=\lim\limits_{h\to0^+}\frac{1}{h}\left(\int_a^{x+h}f(t+h)d(t+h)-\int_a^x f(t+h)d(t+h)\right)-\lim\limits_{h\to0^+}\frac{1}{h}\left(\int_a^{a+h}f(t)dt-\int_a^a f(t)dt\right)$

令 $F(x)=\int_a^x f(t+h)d(t+h)$，上式 $=\lim\limits_{h\to0^+}\frac{F(x+h)-F(x)}{h}-\lim\limits_{h\to0^+}\frac{F(a+h)-F(b)}{h}$

由微积分基本定理，$\because f\in C[a,b]$，$\therefore F$ 在 $[a,b]$ 可导. 且 $F'(x)=f(x)$.

根据 newton-lebniz 公式. 左式 $=\lim\limits_{h\to0^+}\frac{F(x+h)-F(x)}{h}-\lim\limits_{h\to0^+}\frac{F(a+h)-F(a)}{h}$

$=F'(x)-F'(a)=f(x)-f(a)$

16. 令 $F(x)=\int_a^x f(t)d(t)$.

$\because f(x)\in C[a,b]$ $\therefore$ $F(x)$ 在 $[a,b]$ 可导. 又 $\int_a^b f(x)dx=0$

$\because F(b)=F(a)=0$, $\therefore \frac{F(b)}{b}=\frac{F(a)}{a}$

由罗尔中值定理，$\exists\xi\in(a,b)$ s.t. $\left(\frac{F(\xi)}{\xi}\right)'=0$

$\therefore \frac{\xi F'(\xi)-F(\xi)}{\xi^2}=0$ $\therefore F(\xi)=\xi f(\xi)$

即 $\int_a^\xi f(x)dx=\xi f(\xi)$

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系别＿＿＿＿＿ 班级＿＿＿＿ 姓名＿＿＿＿＿＿ 第 页

17 解: $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{(n+1)\cdots(n+n)}}{n}=\lim\limits_{n\to\infty}\sqrt[n]{1+\frac{1}{n}}\sqrt[n]{1+\frac{2}{n}}\cdots\sqrt[n]{1+\frac{n}{n}}$

$=\lim\limits_{n\to\infty}e^{\ln\sqrt[n]{1+\frac{1}{n}}+\cdots+\ln\sqrt[n]{1+\frac{n}{n}}}$ $\because \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\ln(1+\frac{1}{k})=\int_0^1\ln(1+x)dx$

且 $\int\ln(1+x)dx=x\ln(1+x)dx-\int\frac{x}{x+1}dx=(x+1)\ln(x+1)-x$

$\therefore \int_0^1\ln(1+x)dx=2\ln2-1$ $\therefore$ 原式 $=e^{2\ln2-1}$

18.

(1). $\int_n^{n^2}\frac{\sin x}{x}dx=-\frac{\cos x}{x}\Big|_n^{n^2}+\int_n^{n^2}\cos x(-\frac{1}{x^2})dx$

$=\frac{\cos n}{n}-\frac{\cos n^2}{n^2}+\cos\xi\int_n^{n^2}(-\frac{1}{x^2})dx$

$=\frac{\cos n}{n}-\frac{\cos n^2}{n^2}+\cos\xi\cdot\frac{1}{x}\Big|_n^{n^2}$

$=\frac{1}{n}(\cos n-\cos\xi)+\frac{1}{n^2}(\cos\xi-\cos n^2)\to0$

(2). 对于函数 $f(x)=\int_0^x(1-t^2)^n dt$.

$f'(x)=(1-t^2)^n\geq0$ $\therefore f(x)$ 关于 x 递增.

$\therefore$ 对 $\forall\varepsilon>0$. $\int_0^1(1-t^2)^n dt\geq\int_0^{1-\varepsilon}(1-t^2)^n dt=(1-\varepsilon)(1-\eta^2)^n$ $(\eta\in(0,1-\varepsilon))$.

$\geq(1-\varepsilon)[1-(1-\varepsilon)^2]^n$ 调整 ε 大小, 使 $1-\varepsilon<\delta$. 取此时 $\varepsilon=\varepsilon_0$

$\therefore \dfrac{\int_\delta^1(1-t^2)^n dt}{\int_0^1(1-t^2)^n dt}=\dfrac{(1-\delta)(1-\xi^2)^n}{\int_0^1(1-t^2)^n dt}\leq\dfrac{(1-\delta)(1-\delta^2)^n}{(1-\varepsilon_0)[1-(1-\varepsilon_0)^2]^n}<\dfrac{1-\delta}{1-\varepsilon_0}\cdot\delta^n$ $(0<\delta<1)$

当 $n\to\infty$ 时. $\frac{1-\delta}{1-\varepsilon_0}\cdot\delta^n\to0$

不难看有原式 ≥0. $\therefore$ 原式 $\to0$

19. 证:

记 $x=nT+r(x)$. $n\in N$, $r(x)\in[0,T)$

$$\therefore \frac{1}{x}\int_0^x f(t)dt=\frac{1}{nT+r(x)}\int_0^{nT+r(x)} f(x)dx$$

$$=\frac{1}{nT+r(x)}\cdot n\int_0^T f(x)dx+\frac{1}{nT+r(x)}\int_0^{r(x)} f(x)dx$$

$$=\frac{1}{T+\frac{r(x)}{n}}\cdot\int_0^T f(x)dx+\frac{1}{nT+r(x)}\int_0^{r(x)} f(x)dx$$

当 $n\to\infty$ 时, $\frac{r(x)}{n}\le\frac{T}{n}\to 0$

$$\therefore \frac{1}{T+\frac{r(x)}{n}}\int_0^T f(x)dx\to\frac{1}{T}\int_0^T f(x)dx$$

$\because r(x)\in[0,T)$, $f(x)\in C[0,r(x)]$

$$\therefore \left|\int_0^{r(x)} f(x)dx\right|\le M$$

$$\therefore \frac{1}{nT+r(x)}\int_0^{r(x)} f(x)dx\le\frac{M}{nT}\to 0$$

$$\therefore \lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt=\lim_{n\to\infty}\frac{1}{nT+r(x)}\left(n\int_0^T f(x)dx+\int_0^{r(x)} f(x)dx\right)$$

$$=\frac{1}{T}\int_0^T f(x)dx.$$

20.

$\because f \in C[0,+\infty)$，且 $\lim\limits_{x\to+\infty} f(x)=A$．$\therefore$ 对 $\forall \varepsilon>0$，$\exists X$，对 $\forall x>X$，$|f(x)-A|<\varepsilon$

令 $t=nx$

$\therefore \lim\limits_{n\to\infty}\int_0^1 f(nx)\,dx=\lim\limits_{n\to\infty}\frac{1}{n}\int_0^n f(t)\,dt$

要证 $\lim\limits_{n\to\infty}\int_0^1 f(nx)dx=A$．即证：对 $\forall\varepsilon>0$，$\exists N$．对 $\forall n>N$．$\left|\frac{1}{n}\int_0^n[f(t)-A]dt\right|<\varepsilon$

注意到 $\left|\frac{1}{n}\int_0^n[f(t)-A]\,dt\right| \le \frac{1}{n}\int_0^n|f(t)-A|\,dt$

$=\frac{1}{n}\int_0^X|f(t)-A|dt+\frac{1}{n}\int_X^n|f(t)-A|dt$

① $\because X$ 是已知正实数，故 $|f(t)-A|$ 在 $[0,X]$ 有界．设 $M=\sup\limits_{0\le t\le X}|f(t)-A|$

$\therefore \frac{1}{n}\int_0^X|f(t)-A|dt \le M\cdot\frac{X}{n}$ $\quad\because \frac{1}{n}\to 0\ (n\to\infty)$

$\therefore \exists k$，$n>k$ 时．有 $\frac{1}{n}\int_0^X|f(t)-A|dt\le\frac{MX}{n}<\varepsilon$

② $\frac{1}{n}\int_X^n|f(t)-A|dt < \frac{1}{n}\int_X^n\varepsilon\,dt=\frac{n-X}{n}\varepsilon<\varepsilon$

$\therefore$ 原式 $<\varepsilon+\varepsilon=2\varepsilon$

取 $N=k$，对 $\forall\varepsilon>0$、$\exists N$，$n>N$ 时．$\left|\frac{1}{n}\int_0^n f(t)dt-A\right|<2\varepsilon$．

得证 $\lim\limits_{n\to\infty}\int_0^1 f(nx)dx=A$．

21. $x\in(0,\frac{\pi}{4})$时 $\tan x\in(0,1)$ $\therefore \tan^{n+2}x\le\tan^n x\le\tan^{n-2}x$

而 $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx \geqslant \int_0^{\frac{\pi}{4}}\tan^{n+2}x\,dx$

$=\int_0^{\frac{\pi}{4}}\tan^n x\tan^2 x\,dx=\int_0^{\frac{\pi}{4}}\tan^n x(\sec^2 x-1)dx$

$=\int_0^{\frac{\pi}{4}}\tan^n x\,d\tan x-I_n$

$\therefore 2I_n\geqslant \left.\frac{\tan^{n+1}x}{n+1}\right|_0^{\frac{\pi}{4}}=\frac{1}{n+1}$ $\therefore I_n>\frac{1}{2(n+1)}$

由 $I_n=\int_0^{\frac{\pi}{4}}\tan^n x\,dx\le\int_0^{\frac{\pi}{4}}\tan^{n-2}x\,dx$，同理可证 $I_n<\frac{1}{2n}$.

22. 令 $y=x^2$ $I\triangleq\int_0^{\sqrt{2\pi}}\sin x^2\,dx=\frac{1}{2}\int_0^{2\pi}\frac{\sin y}{\sqrt{y}}dy=\frac{1}{2}\left(\int_0^{\pi}+\int_{\pi}^{2\pi}\right)\frac{\sin y}{\sqrt{y}}dy\triangleq I_1+I_2$

令 $z=y-\pi$. $I_2=\frac{1}{2}\int_{\pi}^{2\pi}\frac{\sin y}{\sqrt{y}}dy=-\frac{1}{2}\int_0^{\pi}\frac{\sin z}{\sqrt{z+\pi}}dz=-\frac{1}{2}\int_0^{\pi}\frac{\sin y}{\sqrt{y+\pi}}dy$

$\therefore I=\frac{1}{2}\int_0^{\pi}\sin y\left(\frac{1}{\sqrt{y}}-\frac{1}{\sqrt{y+\pi}}\right)dy$

$\because x\in(0,\pi)$时 $\sin y\cdot\left(\frac{1}{\sqrt{y}}-\frac{1}{\sqrt{y+\pi}}\right)>0$ $\therefore I>0$

23. $\int_0^{2\pi}f(x)\cos nx\,dx=\left.\frac{1}{n}f(x)\cdot\sin nx\right|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}f'(x)\sin nx\,dx=\frac{1}{n}\int_0^{2\pi}(-f'(x))\sin nx\,dx$

$\because f(x)$在$[0,2\pi]$下凸 $\therefore f'(x)$在$(0,2\pi)$递增.

又 $f'(x)$在$[0,2\pi]$连续 $\therefore f'(x)$在$[0,2\pi]$递增 $\therefore -f'(x)$在$[0,2\pi]$递减

由习题 8A-10. $\int_0^{2\pi}(-f'(x))\sin nx\,dx\geqslant 0$ $\therefore \int_0^{2\pi}f(x)\cos nx\,dx\geqslant 0$

25.

证:

由taylor公式.

$\exists\ \xi_1、\xi_2:\ a<\xi_1<b,\ a<\xi_2<b$

使得: $f(x)=f(a)+f'(\xi_1)(x-a)=f'(\xi_1)(x-a)$

$f(x)=f(b)+f'(\xi_2)(x-b)=f'(\xi_2)(x-b)$

$$\therefore \int_a^b |f(x)|dx = \int_a^{\frac{a+b}{2}}|f(x)|dx+\int_{\frac{a+b}{2}}^b |f(x)|dx$$

$$=\int_a^{\frac{a+b}{2}}|f'(\xi_1)(x-a)|dx+\int_{\frac{a+b}{2}}^b|f'(\xi_2)(x-b)|dx$$

$$=|f'(\xi_1)|\int_a^{\frac{a+b}{2}}(x-a)dx+|f'(\xi_2)|\int_{\frac{a+b}{2}}^b(b-x)dx$$

$$\leq \max_{a\leq x\leq b}|f'(x)|\frac{(b-a)^2}{8}+\max_{a\leq x\leq b}|f'(x)|\frac{(b-a)^2}{8}$$

$$=\max_{a\leq x\leq b}|f'(x)|\cdot\frac{(b-a)^2}{4}$$

得证