2011 Mathematical Contest in Modeling (MCM) Summary Sheet

Team Control Number

9665

Problem Chosen

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Summary

As is known to all, in half-pipe games, the shape of the half-pipe has some influences on the performances of the snowboarders in many aspects, such as the "vertical air" and the average degree of rotation. It's a big challenge for the snowboard designers to determine the shape of the half-pipe so as to achieve the desired performances. This problem can be attributed to an optimization problem.

For Question One, an interesting **optimization model** is constructed for the "vertical air" based on the theory of Dynamics. In particular, in order to achieve the maximum vertical distance above the edge of the half-pipe, we have derived the best height of the platform, which is 6.241 meters. On the other hand, by the relationship between the arc angle of the transitions and the vertical component of velocity, the best value for the arc angle of the transitions is determined to be 51.66°. Furthermore, according to the analytic expression of the **curve of velocity recovery**, we calculate the width of the flat is 19.18 meters. Finally, by the four steps method, the length of the U-shape bowl is calculated as 159.95 meters.

For Question Two, the **mechanical characteristics model** is constructed based on the theory of Sport Biomechanics. Particularly, we firstly derived the relationship between the torque and the horizontal twist angle. Then, the function relationship between the horizontal twist angle and the arc angle of the transitions is examined. Also, the situation of twist in the vertical direction can be discussed similarly and finally, in order to maximize the twist in the air, the best values of the arc angle of the transitions, the length and width of the U-shape bowl and the height of the platform are respectively calculated as 45°, 165.96 meters, 19.18 meters and 6.241 meters.

For Question Three, the **weighted tradeoff model** based on the "vertical air" and the twist angle is developed which ensures that the athlete would get the best scores. Based on the relationship among the "vertical air", the twist angle and the corresponding scores obtained, we derived the appropriate weight coefficient and develop a practical course.

Moreover, the influences of other two different shapes of half-pipes on the "vertical air" and twist angle are also discussed. The model is tested by a numerical example and the **sensitivity analysis** about the initial velocity is performed. Finally, based on the analysis about the strengths and weaknesses of the proposed model, we have also considered the refinement of our model, taking the safety of the athlete and the air resistance into account.

Keywords:
Snowboard Half-Pipe Theory of Dynamics Optimization Mechanical Characteristics

Fly Higher with More Twist: The Optimization of a Half-Pipe

Content

I Introduction

The snowboarding was firstly introduced in the early 1970s and since then, it has become more and more popular all over the world.

However, with the rapid development of modern science and technology, there is no doubt that the sport equipment has played an important role for an athlete to improve the performance.

To meet the needs of skilled snowboarders, manufactures have introduced lots of different designs utilizing different materials and shapes for snowboarder courses, which are also currently known as half-pipes (Brennan, et al, 2003). In order to achieve the desired performance, the key point in determining the shape of a half-pipe is to balance the possible requirements, such as the maximum production of “vertical air” and maximum twist in the air.

The performance of an athlete heavily depends on the total air time, average degree of rotation and the “vertical air”. In fact, these elements are also related to the shape of a half-pipe, especially the transition radius, the height and the length and width of the flat bottom of the half-pipe.

In this paper, the theory of Dynamics and Sport Biomechanics are employed to design the shape of a half-pipe for the purpose of maximizing the vertical distance and the twist in the air.

II Symbol Definitions

Symbol	Definition
h_x	the height of platform
r_x	the radius of transitions
δ	the arc angle of transitions
w_x	the width of flat
θ	the twist angle of athlete in the sky
a_x	The length of platform
Δh	vertical air
μ	the coefficient of sliding friction of the surface of half-pipe

F_N	the sustain force given by half pipe to snowboard an athlete
λ	the angle between gravity and vertical direction of arc plane
S	the slide distance from backing to flat to completing acceleration
r_x	the radius of transitions
v_s	the velocity when athlete making flight part
$v_{\max}$	the biggest velocity before sliding on the transitions
$S_{\max}$	the slid distance during acceleration on the flat
f	the friction applied to snowboard
a_c	the acceleration of athlete on the flat
t_t	the time in the air for athlete
ω	angular velocity of rotation
$\hat{M}_x$	the bending moment about *x axis*

III Solution to Question One

3.1 Half Pipes

A half-pipe is basically a U-shaped bowl that allows riders to move from one wall to the other by making jumps and performing Snowboarding Tricks on each transition (Half-pipe snowboarding).

There is a variety of different kinds of half-pipes utilized internationally and the main differences between them are:

- the bottom region is flat or arched,
- the two side arches of the half-pipe are circular or other arches.

Considering the possible differences mentioned above, we mainly divide the following three kinds of half-pipes:

(1) The bottom region and its two sides are respectively flat and circular. This kind of half-pipe is used in Winter Olympics.

(2) The bottom region and its two sides are respectively arched and circular. This kind of half-pipe is used in previous games.

(3) The bottom region and its two sides are respectively flat and elliptical. This kind of half-pipe is used in some local games.

For a skilled snowboarder, it's obvious that the shape of a half-pipe will affect his achievement. In the following discussions, we shall develop a mathematical model and focus on analyzing the first kind of half-pipe. In addition, a few designs of some other kinds of half-pipes are also discussed and compared with respect to the influences on the athlete's performances in games.

3.2 The Description of Half-Pipe We Focused

The Halfpipe Schematic drawn in **Figure 1**:

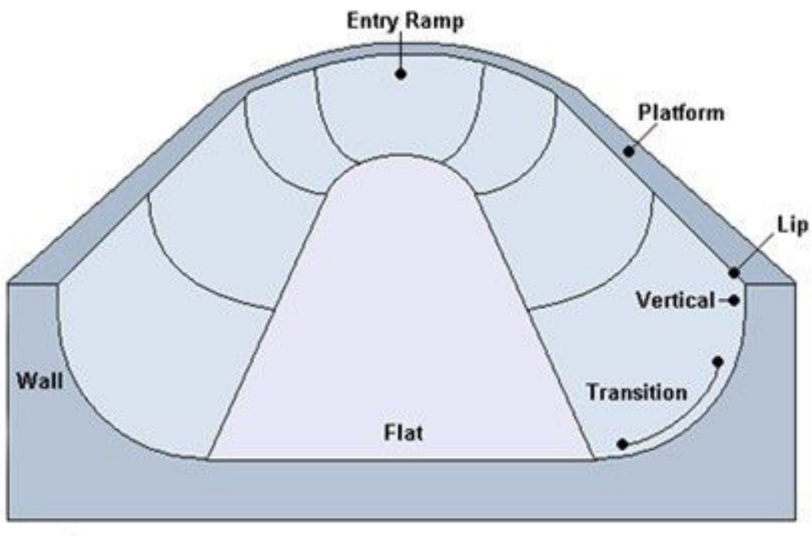

Figure 1. Snowboarding Half-pipe Schematic (Half-pipe snowboarding)
Sources: http://www.abc-of-snowboarding.com/snowboardinghalfpipe.asp

To illustrate the half-pipe more specific, we throw light upon its main elements.

- **Flat**
  Is the center flat floor of the Half-pipe
- **Transitions**
  The curved transition between the horizontal flat and the vertical walls
- **Verticals**
  The vertical parts of the walls between the Lip and the Transitions
- **Platform/Deck**
  The horizontal flat platform on top of the wall
- **Entry Ramp**
  The beginning of the half-pipe where athlete start their run

3.3 Basic Model

In this section, a basic model is developed in order to model the design of a half-pipe and analyze its influence on the performance of an athlete.

3.3.1 Assumptions and Notations

For the sake of convenience of the following discussions, we firstly assume that:

(1) The " vertical air " for an specific athlete is a constant.

(2) The snowboard and snowboarder are an entity and the internal action of them can be ignored.

(3) The air resistance is ignored.

(4) The coefficient of sliding friction of the half pipe surface is fixed.

Now, we are to introduce some notations which will be used in the subsequent Sections.

h_x : denotes the height of platform;

r_x : denotes the radius of transitions;

δ : denotes the arc angle of transitions;

w_x : denotes the width of the flat;

t_x : denotes the thickness of flat;

a_x : denotes the length of U-shape bowl;

b_x : denotes the width of the U-shape bowl.

In particular, a scheme of a half-pipe with the notions introduced above is drawn and presented in **Figure 2**.

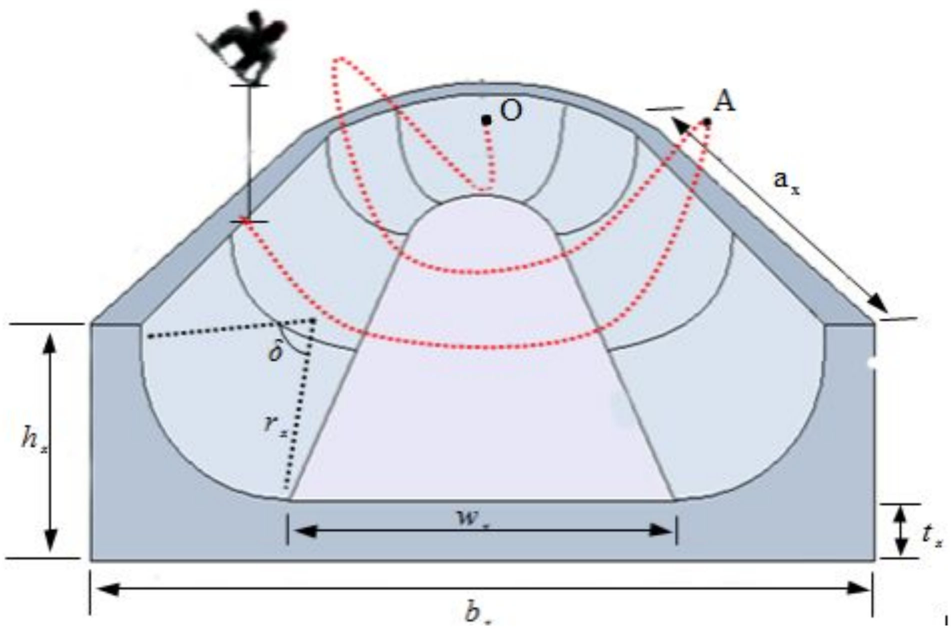

Figure 2. A scheme of a half-pipe with notations.

Based on the discussions above, we have

$$h_x - (r_x - r_x \cdot \cos\delta) = t_x,$$
$$b_x = w_x + 2r_x \cdot \sin\delta + 2t_x.$$

Obviously, in order to complete the design of a half-pipe, we should find the numerical solutions of each variable defined in **Figure 2**.

3.3.2 The Analysis of Snowboarder's Movement

As shown in **Figure 2,** when an athlete starts from point O to a random extreme point A, there only exists the gravity and friction which are in work in the whole process.

According to the law of conservation of energy, we can get

$$\begin{cases} W_p - W_f = mg \cdot \Delta h, \\ \Delta h = h - h_x, \end{cases}$$

where W_p denotes the work done by the snowboarder on the flat and it is used for acceleration, W_f denotes the work done by friction.

Obviously, in order to maximize the "vertical air", we should maximize W_p and minimize W_f, respectively. Note that the resistant function of friction for the snowboard is much smaller than the work done by the snowboarder, the achievement of maximizing the "vertical air" can approximately be obtained only by maximizing W_p.

3.3.3 The Calculation of W_f

Firstly, it's known that the work done by friction is given by $W_f = f \cdot l = \mu \cdot F_N \cdot l$. In order to compute the force F_N, we consider the following two different situations:

- When the snowboarder is on the flat, then we have $F_N = mg$, where m is the total weight including both the snowboard and the athlete.
- When the snowboarder is on the transition, then the centripetal force is provided by the support force of arch and the gravity. Thus, we have

$$F_N - mg \cdot \cos\lambda = \frac{mv^2}{2},$$

where v is the velocity, λ ($0 < \lambda < \delta$) is the angle between the gravity and the vertical direction of arch. We can also get that $dl = r_x \cdot d\lambda$ and the force analysis of the snowboarder is presented in **Figure 3**.

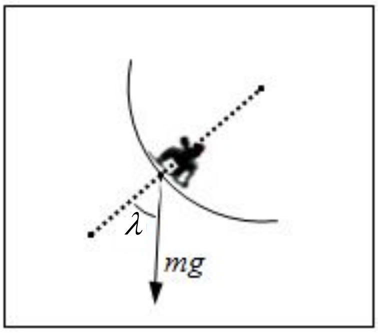

Figure 3. Force analysis of the snowboarder.

Assume that the snowboarder has completed n times flight actions, we can approximately calculate the distance l by the following equation:

$$l = n \cdot (w_x + 2r \cdot \sigma) / \cos\alpha .$$

Thus, we can get

$$W_f = \frac{\mu \cdot nmg \cdot w_x}{\cos\alpha} + \mu \int_0^{\delta} n(mg\cos\lambda + \frac{mv^2}{r_x}) \cdot 2r_x \cdot d\lambda .$$

Note that it is difficult to deal with the friction changes of the transitions and the work done by the friction is much less than that done by the athlete used for acceleration, we approximate the value of the friction of the transitions as μmg and it will not induce a big difference. So, we have

$$W_f = \mu \cdot nmg \cdot (w_x + 2r \cdot \delta) / \cos\alpha .$$

3.3.4 The Discussion of W_p

Snowboarders get higher velocity by stomping or rising leg on the flat. However, as to friction of half-pipe, the velocity has a maximum value $v_{\max}$. When the velocity achieves $v_{\max}$, snowboarder can't improve it by snowboarding skills. We assume the work done by snowboarder when he accelerate for i th on the flat as W_{pi} . However, in the process of design the half-pipe, we don't care about the numerical solution of W_{pi} . The reason of discussing it is to ensure the integrity and accuracy of our model.

Combing with the equations derived above, we obtain

$$\Delta h = \frac{\sum_{i=1}^{n} W_{pi} - \mu nmg(w_x + 2r\delta)/\cos\alpha}{mg}.$$

Obviously, the value of Δh is decided by W_p directly. For the process of every slid motion done by athlete is same basically, we can calculate W_p by

$$W_p = \frac{1}{n} \cdot \sum_{i=1}^{n} W_{pi}$$

By analyzing the above equations, we can get that the “vertical air” is maximum if the velocity when athlete make flight action is limit value.

3.3.5 The Function of W_p

W_p is the work done by athlete used for accelerating and it is used to compensate the lose caused by the work done by friction.

Figure 4 shows the specific situation we investigate.

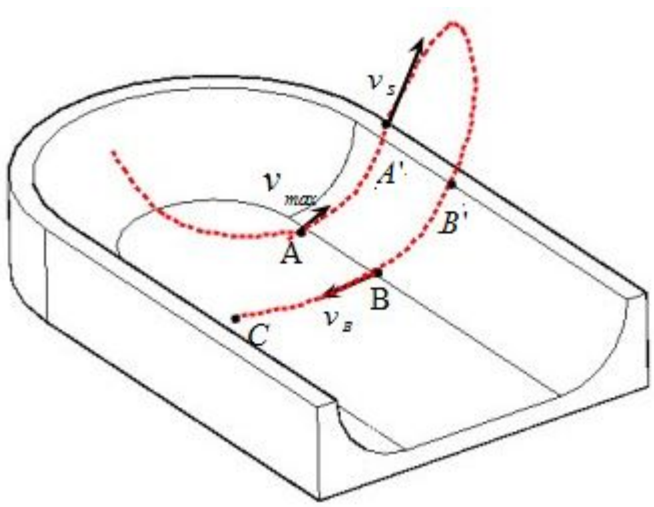

Figure 4. The scheme of slid motion done by athlete

As is shown in **Figure 4**, from point A to B, the velocity of athlete decreases because of the friction. In this process, the following equations are obeyed.

$$\begin{cases} \frac{1}{2}mv_{\max}^2 = \frac{1}{2}mv_B^2 + W_{f_1} \\ W_{f_1} = f_1 \cdot (AA' + BB') \end{cases}.$$

From point B to C, the velocity recoveries to the maximum value with the help of

W_p . So we have,

$$\begin{cases} \frac{1}{2}mv_{\max}^2 = \frac{1}{2}mv_B^2 - W_{f_2} + W_p \\ W_{f_2} = f_2 \cdot BC \end{cases},$$

where, f_1 is the friction applied to snowboard of AA' and BB', f_2 is that of BC and W_{f_1}, W_{f_2} are the corresponding work done by f_1 and f_2. Thus, we have

$$W_p = W_{f_1} + W_{f_2}$$

3.3.6 Approach Parameters of Half-Pipe

The height of the platform and the radius of transition

The height of platform is related to the original velocity of athletes. In the process of sliding down from the entrance, there is almost no loss of energy, namely, the gravitational potential energy translates into kinetic energy totally. So we have

$$mg \cdot h_x = \frac{1}{2}mv_0^2$$

The original velocity decides the achievement of athletes in competition and is related to the “vertical air”. Thus , it is very important to consider it in the design of the height of platform.

According the study by Huang and his team, in order to get a relative good score, athlete should ensure the original velocity when he or she enters the half-pipe to be $11.06m/s$ at least.

Since the abusive high of platform will result in the big loss of velocity before the flight part made, the relative suitable original velocity for design platform is the least one. Thus,

$$h_x = \frac{v_0^2}{2g} = 6.241 \text{ m}.$$

The arc angle of transitions

Athletes make flight part on the top of transitions and his velocity of flight can be resolute in the three directions-x, y and z. Namely,

$$v_x = v_s \cdot \cos\alpha_x,\ v_y = v_s \cdot \cos\alpha_y \text{ and } v_z = v_s \cdot \cos\alpha_z,$$

where, v_x, v_y and v_z are the component velocities of v_s in the directions of x, y and z respectively and α_x, α_y and α_z are the angle of v_x, v_y and v_z. **Figure 5** shows the specific situation.

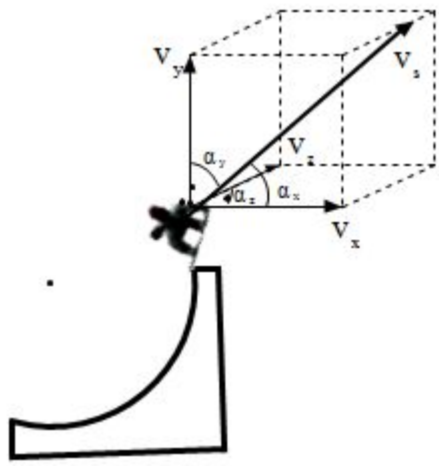

Figure 5. The scheme of resolution of v_s

What is more, in vertical plane, just like what is shown in **Figure 6** , the relationship of δ, α_x and α_y obeys the following equations approximately.

$$\alpha_x \approx \delta \text{ and } \alpha_y \approx 90^\circ - \delta$$

Thus,

$$v_x = v_s \cdot \cos\delta \text{ and } v_y = v_s \cdot \sin\delta .$$

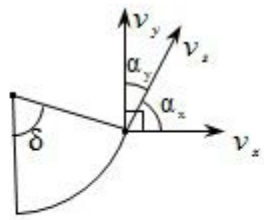

Figure 6. The scheme of component velocities in vertical plane.

The height of flight Δh increase with the vertical component velocity v_y. As is shown in **Figure 4**, from point A to A', the relationship of energy obeys

$$\frac{m \cdot {v_s}^2}{2} = \frac{1}{2} \cdot m \cdot v_{\max}^2 - f_1 \cdot AA' - mg \cdot h_x .$$

Namely,

$$v_s = \sqrt{\frac{2}{m}\left(\frac{1}{2}mv_{\max}^2 - f_1\left(\frac{w_x + r_x\delta}{\cos\alpha}\right) - mgr_x(1-\cos\delta)\right)}.$$

Then, we can get

$$v_y = \sqrt{\frac{2}{m}\left(\frac{1}{2}mv_{\max}^2 - f_1\left(\frac{w_x + r_x\delta}{\cos\alpha}\right) - mgr_x(1-\cos\delta)\right)} \cdot \sin\delta$$

Where f_1 is the friction force in the inclined plane, which decrease with δ. For the variation of it is not significant, we regard it as a constant.

We assume that $m = 75kg$, $\mu = 0.1$ and $\alpha = 5^\circ$, then the solution of δ is transformed as a question which calculating the maximum of function of one variable.

Utilizing the MATLAB software, we can get the value of δ corresponding to the max component velocity in vertical direction is 51.66° .

The relationship of δ and v_y is shown in **Figure 7**

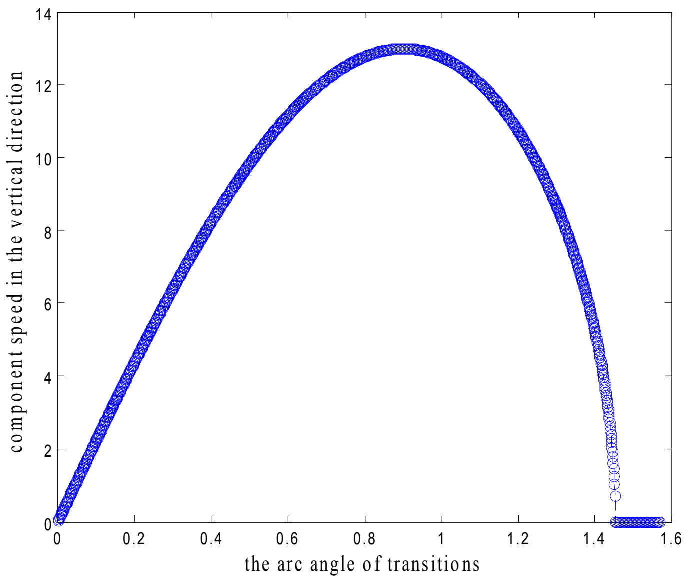

Figure 7 .The relationship of δ and v_y

Just as shown in **Figure7**, the relationship of two variables obeys parabola approximately. When $\delta = 51.66^\circ$, the v_y reaches the maximum value.

According the equation

$$v_y^2 = 2g \cdot \Delta h,$$

we can derive the "vertical air" is maximal when $\delta = 51.66^\circ$. So, the radius of transitions can be calculated as

$$r_x = \frac{h_x}{1-\cos\delta} = 16.44\ m.$$

The width of flat

The process of athlete doing work for acceleration can't be completed in a very short distance. It must reach the minimum distance S_{max} for accelerating velocity to v_{max}. Base on this, we promote **the curve of velocity recovery**.

Just as shown in **Figure 8**, the curve has following characteristics:

(1) Horizontal axis presents the slid distance to the start point, and vertical axis presents the maximum velocity can be achieved corresponding to a fixed slid distance.
(2) The recovery function to velocity decreases with the increment of slid distance S, namely the slope in figure decreases with S.
(3) In order to obtain v_{max}, S_{max} should be slid at least.
(4) If the slid distance is longer than S_{max}, the achieved velocity at last can be regarded as a constant approximately.

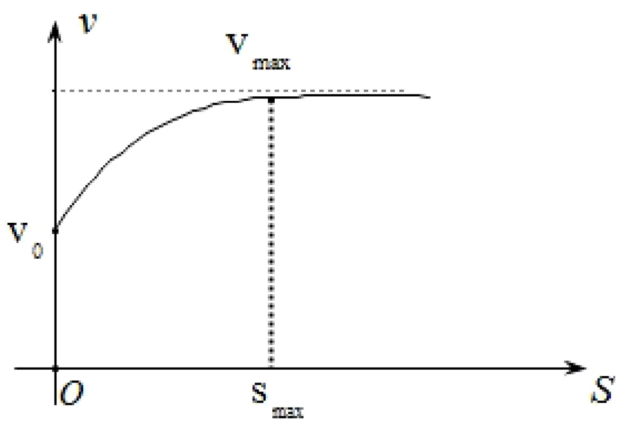

Figure 8. The curve of velocity recovery

Fortunately, Harding, et al. (2007) investigate the relationship between the acceleration and the time used for sliding by utilizing a sliding Fast Fourier Transform (FFT) window and subsequent power analysis and characterize it in **Figure 9** .

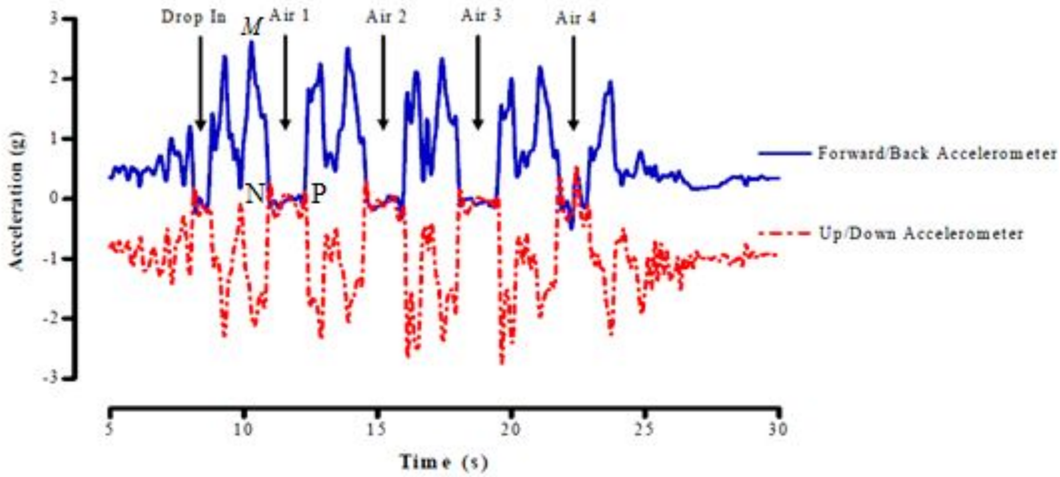

Figure 9. A unfiltered accelerometer trance associated with a snowboarding run

In **Figure 9,** the point M characterizes the time when athlete attach flat after leaving transitions. The acceleration is very big at the moment of M. The point N presents the slid distance of an athlete reaches $S_{\max}$ and the velocity is max. The point P characterizes the time when athlete attach the other transition after completing the slide on flat.

We can get, by analyzing, the conclusion that the velocity of athlete will not change after acceleration for some time in the condition of the length of flat is long enough. To investigate it more deeply, we distill the relationship in the figure into a approximate broken line which shown in **Figure 10**.

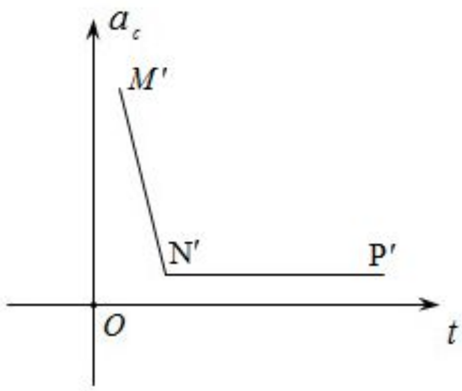

Figure 10 .The distilled relationship between acceleration and time

Combing the data shown in **Figure 9**, The relationship between the acceleration and time when athlete accelerate in flat can be characterized by the following equation.

$$a_c = \begin{cases} 2.5g - 2.5gt, & \text{t} \le 1\text{s} \\ 0 & \text{t>1s} \end{cases}.$$

Thus, the velocity of athlete at random can be calculated by

$$v = v_1 + \int_0^1 (2.5g - 2.5gt)dt = v_1 + 2.5gt - 1.25gt^2 .$$

The slid distance when the acceleration decreased to 0 at the time of $t = 1s$ is $S_{\max}$, which can be calculated by

$$S_{\max} = \int_0^1 (v_1 + 2.5gt - 1.25gt^2)dt .$$

Note that $v_1 = 11.06\ m/s$, we get $S_{\max} = 19.27m$.

In addition, combing the discussion above and the assumption that $\alpha = 5^\circ$, we get

$$w_x = S_{\max} \cdot \cos\alpha = 19.18m$$

The solution of the length of a half-pipe

The length of a half-pipe is decided by the slid distance between two continuous fight part and the times of flight parts made by athlete simultaneously. Thus,

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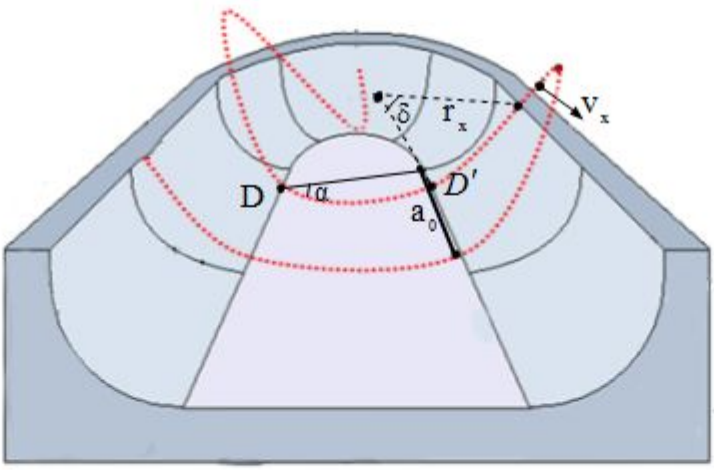

Figure 11. The scheme of process of sliding

From **Figure 11**, we can get that

$$a_0 = DD' \cdot \sin\alpha + 2r_x \cdot \delta \cdot \sin\alpha + v_x \cdot t_t .$$

In order to get the numerical solution of a_x, the following 4 steps should be applied.

Step1: Calculating v_s

From **Figure 4**, we have

$$\frac{1}{2}mv_s^2 = \frac{1}{2}mv_{\max}^2 - f_1 \cdot AA' - mgh_x .$$

Thus,

$$v_s = \sqrt{\frac{2}{m}(\frac{1}{2}mv_{\max}^2 - \mu mg \cdot AA' - mgh_x)} .$$

In order to calculate v_s, the parameters m, $v_{\max}$, AA' and v_s should be known.

We assume that m is 75kg which is the average weight of a snowboarder.

Then,

$$v_{\max} = (v_1 + 2.5gt - 1.25gt^2)|_{t=1} = 23.31\ m/s .$$

Note that the range of coefficient of sliding friction is from 0.03 to 0.2, we assume that it is 0.1. From **Figure 4**, we can also get that

$$AA' = \frac{r_x \cdot \delta}{\cos\alpha} .$$

Thus, v_s can be calculated.

Step2: The solution of v_x and t_t by utilizing v_s

$$v_x = v_s \cdot \cos\alpha_x = v_s \cdot \cos\delta,\quad v_y = v_s \cdot \sin\delta \quad \text{and} \quad t_t = 2\frac{v_y}{g}$$

Step3: The solution of DD'

$$DD' = \frac{w_x}{\cos\alpha}$$

Step4: Getting a_x by calculation of a_0

Combing the above derivation, we can get that

$$a_0 = 22.85 \text{ m}$$

According the current investigations, the times of flight part for a athlete is about 7. Therefore, we have

$$a_x = 159.95m$$

IVMechanical Characteristics Model of Question Two

4.1 Assumptions

- Athlete can transform the horizontal velocity to the one twist with the central axis of snowboarder.
- The path of sliding and the “ vertical air” are unchangeable
- The velocity of athlete can reach $v_{\max}$

4.2 Torque

In order to maxim twist in the air, athlete must bend his body. A torque, which is perpendicular to snowboard’s axis of rotation, is generated simultaneously when snowboarder bending his body. **Figure 12** describe the specific situation.

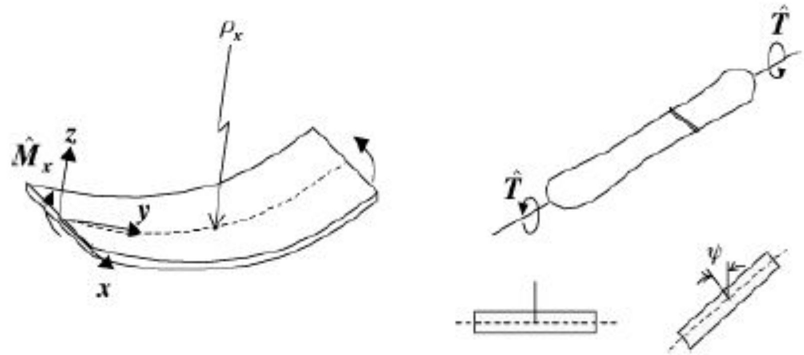

Figure 12. Torque applied to the snowboard and the angle of twist
Source: Brennan, et al. Modelling the mechanical characteristics and on-snow performance of snow boards.

4.3Mechanical characteristics

We treat the snowboard as a thin beam and relate the applied moment and applied torque to the radius of curvature and the rate of twist by the relationship (Kollár & Springer 2003):

$$\begin{Bmatrix} 1/\rho_x \\ \theta \end{Bmatrix} = \begin{bmatrix} W_{22} & W_{24} \\ W_{24} & W_{44} \end{bmatrix} \begin{Bmatrix} \hat{M}_x \\ \hat{T} \end{Bmatrix},$$

where $\hat{M}_x$ is the bending moment about the *x* axis, $1/\rho_x$ is the curvature of the longitudinal *y* axis (**Figure 4**), $\hat{T}$ is the applied torque, ψ is the rate of twist which is related to the angle of twist θ by the expression: $\psi = \frac{d\theta}{dy}$ and W_{22} W_{24} and W_{44} are the elements of the compliance matrix. The origin of the local $x-y-z$ co-ordinate system is at the centroid（Brennan, et al，2003）.

When only a bending moment is applied we have:

$$\frac{1}{\rho_x} = W_{22}\hat{M}_x \text{ and } \theta = W_{24}\hat{M}_x .$$

Since, W_{24} is decided by the intrinsic property of half pipe. we only need to calculate $\hat{M}_x$. Note that $\hat{M}_x$ is the product of M_u (the mass of half pipe) and the projection of transition arc at the direction of x. Namely,

$$\hat{M} = M_u \cdot r_x \cdot \sin\delta ,$$

where r_x is determined by δ. Thus, in order to maxim the angle of twist, we should maximize δ and can get that

$$\theta = \int W_{24}\hat{M}_x dy$$

4.4 The Determination of Parameters of Half-Pipe

In order to maximize the twist angle, the bending moment $\hat{M}_x$ should be maximized. The method of determination of parameters of half-pipe is similar to that of question 1.

Athlete accelerates his velocity to $v_{\max}$ on the flat, slides to transition at this velocity and makes flight part. The key of ensuring the maximum of twist angle of athlete is design of height and arc of transition.

We resolute the velocity v_s when athlete begins his flight part, and get:

$$v_x = v_s \cdot \cos\alpha_x ,\ v_y = v_s \cdot \cos\alpha_y \text{ and } v_z = v_s \cdot \cos\alpha_z .$$

After the flight part, v_x and v_z are unchangeable, and athlete can convert v_x into the rotational velocity (v_x') whose central axis is that of snowboard. So, $v_x' = \chi v_x$, where χ is the efficiency of converting and only related to the competence of athlete. In addition, with the consideration of benefitting athlete to return back snowboarding run, α_z should be relative big.

Since the radius of rotation after athlete making fight part is $\frac{l_0}{2}$, we can get the angular velocity of rotation ω ($\omega = v_x' / (l_0 / 2)$). In addition, the total time for athlete flight t_t ($t_t = 2v_y / g$) can be calculated.

Thus, the angle of twist could be calculated as

$$\theta = \omega \cdot t_t$$

For $\alpha_x \approx \delta$ and $\alpha_y \approx 90^\circ - \delta$, we can get

$$v_x = v_s \cdot \cos\delta \text{ and } v_y = v_s \cdot \sin\delta .$$

Thus, we can derive that

$$\theta = \omega \cdot t_t = \chi \cdot v_s^2 \cdot \sin\delta \cdot \cos\delta = \frac{\chi \cdot v_s^2 \cdot \sin 2\delta}{2} .$$

Based on the above investigation, we can get the following conclusion.

When the arc angle of transition δ equals 45 degree, the twist angle is maximal.

Regarding twist angle as an optimizing object and by the method used in question one, we obtain

$$h_x = 6.241 \text{ m} .$$

And the radius of transitions can be calculated as

$$r_x = h_x / (1 - \cos\delta) = 21.3\ m$$

The method used in calculating the width of half pipe is similar to the one used in question one. However, the arch angles of transitions in two methods are different. The numerical solution is $a_x = 165.96m$.

V Tradeoffs Model to Question Three

5.1 The Determination of Parameters

Base on analysis we have done, we can conclude that the parameters calculated with different objectives- "vertical air" or twist in the air-are different. In order to design the half-pipe which is most suitable to athletes, we tradeoff the two kinds of objectives and get the corresponding parameters.

Objective of optimization: the best achievement of athlete in the competition.

Harding and his team (2008) provide the relationship among the "vertical air", degree of rotation and the predicted scores of athlete obeys the following equation.

$$PS = 3.42AVA + 0.011ADR - 1.794\,,$$

where PS is the predicted scores of athletes in the competition, AVA is the average "vertical air" of athletes, and ADR is the average degree of rotation.

Basing on the above equation, we can find that the "vertical air" is much more important than the degree rotation for the scores of athletes.

For the objective of optimization is the scores given by judges, we regard the coefficients of AVA and ADR as weight when designing half-pipe.

The weight of "vertical air" is

$$k_1 = \frac{3.421}{3.421 + 0.011} = 0.997\,.$$

And , the weight of degree of rotation is

$$k_2 = \frac{0.011}{3.421 + 0.011} = 0.003\,.$$

As the width of flat and the height of platform are equal in the calculations with different objective, there is no need to weight them. Namely,

$$h_x = 6.241m \text{ and } w_x = 19.18m$$

To the remain parameters, a_x and r_x can be decided by δ, h_x and w_x. Thus, in order to complete the design of half-pipe in the tradeoff model, the only things to do is to determine the value of δ.

According to the principle of weighting, the most suitable arc angle of transitions δ_t should be calculated by

$$\delta_t = k_1\delta_1 + k_2\delta_2 = 51.63^\circ\,,$$

where the δ_1 and δ_2 are arc angle of transitions in question one and two respectively.

It is easy to get the following parameters

$$r_x = 16.37m \text{ and } a_x = 160.19m\,.$$

5.2 The Inclined Angle of Half-Pipe's Placement

The inclined angle of half-pipe's placement refers to the angle of flat plane and horizontal one, which is unrelated to the size of half-pipe and only decided the method of placement.

The angle is used for benefitting to capture the high velocity and reducing the friction force. And a multitude of researchers have done the investigation of inclined angle. Combing them, we get a recommended value which is 18° . If the size of half-pipe is relative small, the angle should decrease slightly, however, it should in the range of 15~18° .

VI The Discussion of other Kinds of Half Pipe

6.1 The Half Pipe whose Transitions are Ellipse

The design parameters of this kind half pipe can be applied by two methods, as is shown in **Figure 13**.

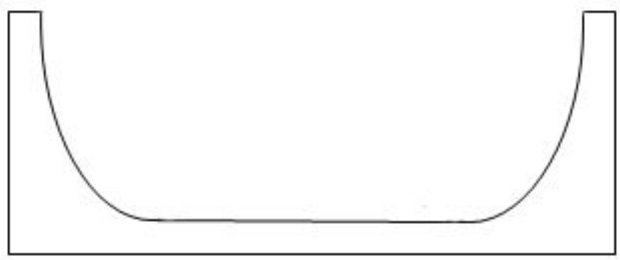

Figure 13 a. The first method for designing half-pipe

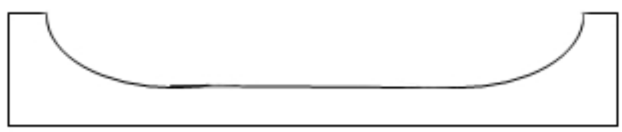

Figure 13 b. The second method for designing half-pipe

To the fist method, for the slope of transition is too steep, it is very hard for athlete sliding to the top which used for making flight part. Even though he or she can reach the top of half pipe, the huge loss on velocity can be ignored too. Thus, the difficulty of snowboarding on this kind of half-pipe is relative big, and it is suit for the snowboarder who pursuit high difficulty.

To the second one, it is equivalent to extend a distance at each sides of flat. In the condition that the width of flat is wide enough, this method will waste the resource of half-pipe. What is more, the arc angle of transitions is difficult to control and it is easy to result in athlete flying out the half-pipe for the reason of the slope of transition is too gentle.

6.2 The Half-Pipe whose Flat is a Arc

The transition of this kind of half-pipe is the same to that of the one we mainly discussed, but, the flat of it is arced. For a variety of reasons such as athlete is more vulnerable when he or she sliding on it for it is difficult to stop on the arc run, the effect of acceleration is not obvious for it rely on the static friction, this kind of half-pipe have been abolished.

Figure 14 shows the shape of this kind of the half-pipe.

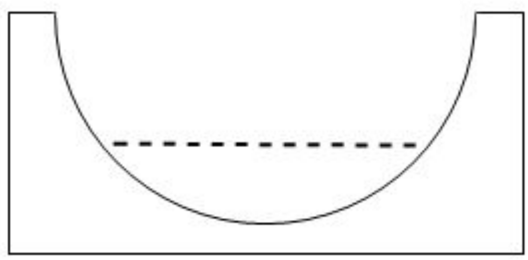

Figure 14. The shape of half-pipe whose flat is arced.

VII The Verification of Our Models

7.1 The Verification by Living Example

We design three kinds of U-shape half-pipes, and their parameters are shown in **Table 1**. In order to compare the difference between the current half-pipes used in the competition and the one we design, we collect 6 sets of data which character the parameters of 6 different kinds of practical half-pipe, see **Table 2**.

Table 1. The parameters of half-pipes we design unit (m)

Parameters	Half-pipe in question 1	Half-pipe in question 2	Half-pipe in question 3	Unit
The arc angle of transitions	51.66	45	51.63	degree
The length of half-pipe	159.95	165.96	160.19	m
The width of flat	19.18	19.18	19.18	m
The height of platform	6.241	6.241	6.241	m

Table 2. The parameters of current half-pipes unit (m)

Parameters	Data appeared in different reference				
The length of half-pipe	160-220	160	80	100-110	160-200
The width of flat	18	18	18	13-15	18
The height of platform	7~8	6	5	3-3.5	5-6

By **Table 1**, we can find, obviously, our half-pipe belongs to the relative big one. This is because of the original velocity used for designing the height of platform is that of athlete who competing in big games. Thus, the half-pipe decided by the parameters has a relative big size and can be used for big competitions, such as Winter Olympics.

Combing the analysis of the two tables, the parameters of our half-pipes are close to that of current big one. And this phenomenon can verify the correctness of our model.

7.2 Sensitivity Analysis

By changing the original velocity used for designing h_x and utilizing our model, we calculating a set of parameters of half-pipe. The range of original velocity is from 7m/s to 12m/s, and we take 11 velocities totally with the interval of 0.5 m/s. The data of parameters with different velocity is drawn in **Figure 15**.

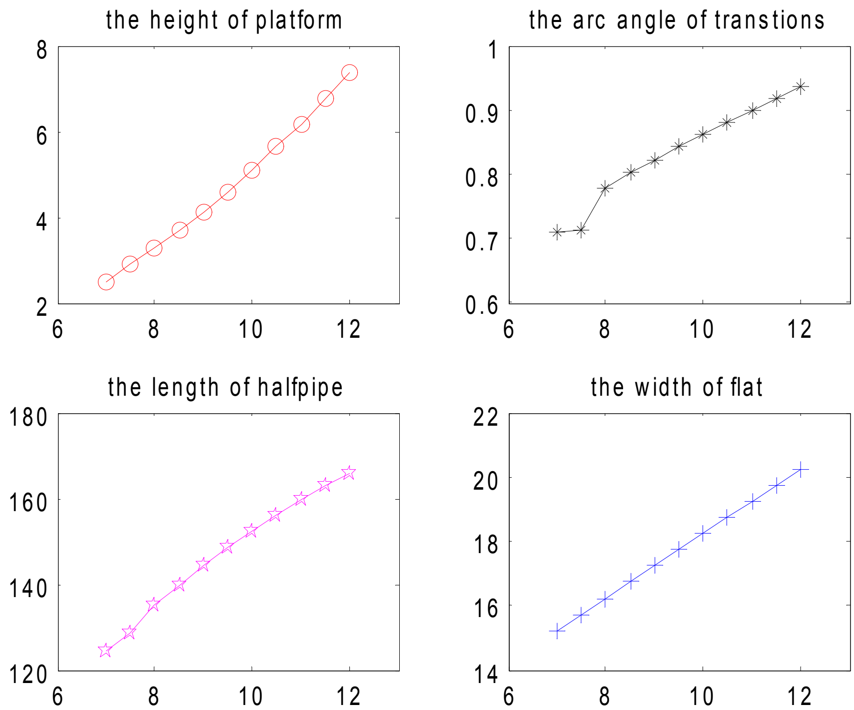

Figure 15 The trend of half-pipe parameters in sensitivity analysis

As **Figure 15** showing, we can get that

- The size of half-pipe increase with v_0
- h_x, a_x and w_x are linear with v_0 approximately
- The arc angle of transitions has the broken line relationship with v_0, and the linearity is better when v_0 is relatively big.

The specific data of them are shown in **Table 3**

Table 3. The data calculated in sensitivity analysis

v_0 m/s	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5	12
a_x m	124.31	128.72	135.12	139.78	144.25	148.47	152.45	156.15	159.1	162.94	165.94
h_x m	2.50	2.89	3.27	3.69	4.13	4.60	5.10	5.63	6.17	6.75	7.35
w_x m	15.22	15.73	16.23	16.73	17.23	17.73	18.24	18.74	19.24	19.74	20.24
δ degree	40.69	40.79	44.67	46.03	47.17	48.29	49.39	50.49	51.58	52.64	53.70

From **Table 3**, when v_0 takes the value of 8, 9, 10 and 11m/s, the theoretical results match the current half-pipe very well, which agrees with our model.

Base on the data in **Table 3**, we divide the half pipes into three kinds-small, medium and big and recommend the relative classic parameters of them.

The **big half-pipe:**

$$a_x = 159.71,\ h_x = 6.17,\ w_x = 19.24 \text{ and } \delta = 51.58.$$

The **medium half-pipe:**

$$a_x = 139.78,\ h_x = 3.69,\ w_x = 16.73 \text{ and } \delta = 46.03.$$

The **small half-pipe:**

$$a_x = 128.72,\ h_x = 2.89,\ w_x = 15.73 \text{ and } \delta = 40.79.$$

On the condition of other requirement required, the designer can look up **Table 3** to get the corresponding parameters of half-pipe. Obviously, by sensitivity analysis, the range available is extended largely.

VIII Strengths and Weaknesses

8.1 Strengths

- **Simplicity:** The mathematical model we construct is simple enough to understand with a small amount of mathematical skill. And, the calculation of it can solved by simple mathematical software such as Matlab.

- **Flexibility:** Models in this paper consider influence on the performance of athletes given by the parameters-height, width, angle and so on –of half pipe to ensure the flexibility and entity of the design. By doing so, it can design different shape of half pipes with different requirements.

- **Operability:** Our models take into account the criteria of snowboarding and promote the design plan, which helps athlete perform better in the competitions.

- **Developed from history data:** All the data we used are from the records of Olympic Games.

- **Refinement:** Base on considering other factors such as injures of athlete, we refine our model.

8.2 Weaknesses

- **Assumptions:** Simplifying assumptions such as ignoring air resistance had to be made in order to create a calculable model.

- **Inputs:** This verification of model requires a multitude data, some of which is difficult to obtain.

IX The Refinement of our Model

9.1 The Refined Model Considering the Air Resistance

As the equation (Answer.com) of calculating the air resistance is

$$f_a = \frac{1}{2} c \cdot \rho \cdot S_w \cdot v^2$$

where, f_a is the air resistance, c is the coefficient of air resistance, ρ is the density of air. S_w is the frontal area of object and v is relative velocity of object and air. In addition, c and ρ are constants. And S_w is variable, however, the variation is very small.

Based on the analysis above, we can conclude that

In the process of sliding, the air resistance is proportional to the square of velocity. In horizontal direction,

$$\frac{dS}{dt} = v.$$

And the work done by air resistance is

$$\begin{aligned} W_{fa} &= \int_0^{S_{\max}} f_a dS \\ &= \int_0^{S_{\max}} (\frac{c\rho S_w}{2}) \cdot v^2 dS. \end{aligned}$$

The equation of velocity on the flat investigated in question is applicable and $S_{\max}$ is reached when $t = 1s$. Thus,

$$\begin{aligned} W_{fa} &= \int_0^1 (\frac{c\rho S_w}{2}) \cdot v^3 dt = \int_0^1 (\frac{c\rho S_w}{2}) \cdot (v1 + 2.5gt + 1.25gt^2)^3 dt \\ &= \frac{c\rho S_w}{2} \cdot (\frac{2425}{224} g^3 + \frac{95}{8} g^2 + 5{v_1}^2 g + {v_1}^3) \end{aligned}$$

For the given c, S_w, ρ and v_1, the numerical solution of W_{fa} is a constant. Thus, just by substituting the sum of W_f and W_{fa} into the models we construct, the parameters of half-pipe can be calculated.

9.2 The Refined Model Considering the Safety of Athlete

Snowboarders are vulnerable in the competition, and combing the safety factor into the design of half-pipe is necessary. By analyzing the relationship of safety and the twist angle, we can get the function between them. Then, by substituting the factor of safety by the twist angle into the model we constructed, the parameters of half-pipe are also be calculated.

Reference

[1] Brennan,S.M. et al. 2003. Modelling the mechanical characteristics and on-snow performance of snowboards. Sports Engineering 6, 193-206.

[2] ABC-OF-SNOWBOARDING. Halfpipe Snowboarding/ Half Pipes Halfpipes. http://www.abc-of-snowboarding.com/snowboardinghalfpipe.asp.

[3] Hongguang,Yuan.et al.2009.Factors Influencing Velocity Away from Decks in Snowboard Half-pipe.Journal of Shenyang Sport University 28(3),16-28.

[4] Harding,J.W. et al. 2007. Feature Extraction of Performance Variables in Elite Half-Pipe Snowboarding Using Body Mounted Inertial Sensors. http://www.anarchistathlete.com/wp-content/uploads/2009/05/harding-et-al-_2007_-feature-extraction-hp-kpvs-inertial-sesnors.pdf.

[5] Kollár, L.P. and Springer, G.S. 2003. Mechanics of Composite Structures, p. 209. Cambridge University Press, New York, NY, USA.

[6] Answers.com. What is the formula for air resistance please state what each symbol or letter stands for? http://wiki.answers.com/Q/What_is_the_formula_for_air_resistance_please_state_what_each_symbol_or_letter_stands_for.

[7] Harding, J.W and James,D.A.2010. Performance Assessment Innovations for Elite Snowboarding. 8th Conference of the International Sports Engineering Association.

[8] Harding,J.W. et al. 2008. Automated scoring for elite half-pipe snowboard competition: important sporting development or techno distraction? Sports Technology Journal. No.6, 277-290.

[9] Snowboarding. Halfpipe Construction: The Shape of Things to Come. http://snowboarding.transworld.net/1000022552/how-to/halfpipe-construction-the-shape-of-things-to-come/

[10] Wikipedia, the free encyclopedia. Half-pipe, http://en.wikipedia.org/wiki/Halfpipe.

[11] Subic, Aleksandar,et al.2010. Experimental installation for evaluation of snowboard simulated on snow dynamic performance.8th Conference of the International Sports Engineering Association

[12] Buffinton, Keith W.2003.Laboratory, computational and field studies of snowboard dynamics, Sports Engineering 6,129-137.

[13] Thomas, H, et al.2010.Recognizing Turns and Other Snowboarding Activities with a Gyroscope.http://ieeexplore.ieee.org/xpl/mostRecentIssue.jsp?punumber=5648406

Appendix

A 1. The solution of δ in question 1

```
f = @(x)-sqrt(2/75*(0.5*75*23.31^2-0.1*75*9.8*(19.81+25.*x)/cos(5*pi/180)
-75*9.8*25*(1-cos(x)))).*sin(x); % The positive number of vy
x = fminbnd(f, 0, 2)
y = -f(x)
```

A2. The Figure characterizes the relationship between arch angle and velocity

```
syms o;
o=0.001:0.001:pi/2;
g=9.8;a=5*pi/180;rx=25;wx=19.81;
m=75;vmax=23.31;f1=0.1*m*g;
vy=sqrt(2/m*(0.5*m*vmax^2-f1*(wx+rx.*o)/cos(a)-m*g*rx*(1-cos(o)))).*sin(o);
plot(o,vy)
```

A3. The solution of width of half-pipe

```
a=5*pi/180;rx=16.44;o=51.66*pi/180;smax=19.27;rx=16.44;
hx=6.241;u=0.1;m=75;g=9.8;vmax=23.31;wx=19.18;n=7;
AB=rx*o/cos(a);
CD=smax+2*AB;
vs=sqrt(2/m*(0.5*m*vmax^2-u*m*g*AB-m*g*hx));
vx=vs*cos(o);
vy=vs*sin(o);
tt=2*vy/g;
a0=CD*sin(a) +2*rx*o*sin(a) +vx*tt;   % the solution for horizontal distance
ax=a0*n
```

A4. Sensitivity Analysis

```
% The solution for parameters of half-pipe
thta=[40.6857    40.7889    44.6735    46.0314    47.1659...
      48.2889    49.3947    50.4948    51.5777    52.6434    53.6976];
o=thta*pi/180;
vmax=[19.2500    19.7500    20.2500    20.7500    21.2500    21.7500...
         22.2500    22.7500    23.2500    23.7500    24.2500];
wx=[15.2246    15.7265    16.2284    16.7303    17.2322    17.7342...
      18.2361    18.7380    19.2399    19.7418    20.2437];
smax=[15.1667    15.6667    16.1667    16.6667    17.1667    17.6667...
```

```
        18.1667        18.6667       19.1667       19.6667       20.1667];
 hx=[2.5 2.89 3.27 3.69 4.13 4.60 5.1 5.63 6.17 6.75 7.35];
% The parameter matrix corresponding to the chosen data
a=5*pi/180;rx=16.37;
u=0.1;m=75;g=9.8;
AB=rx*o/cos(a);
CD=smax+2*AB;
vs=sqrt(2/m*(0.5*m*vmax.^2-u*m*g*AB-m*g*hx));
vx=vs.*cos(o);
vy=vs.*sin(o);
tt=2*vy/g;
a0=(CD*sin(a) +2*rx*o*sin(a) +vx.*tt);
a1=a0*7
% Plotting the relationship figures of  vy  and parameters of half-pipe
v0=7:0.5:12;

subplot(231)
plot(v0,hx,'r-o')
title('hx'),xlabel('v0')

subplot(232)
plot(v0,smax,'b-d')
title('smax'),xlabel('v0')

subplot(233)
plot(v0,o,'k-*')
title('o'),xlabel('v0')

subplot(234)
plot(v0,a1,'m-p')
title('a1'),xlabel('v0')

subplot(235)
plot(v0,vmax,'K-+')
title('vmax'),xlabel('v0')

subplot(236)
plot(v0,wx,'c-<')
title('wx'),xlabel('v0')
```

A5. The data appeared in references

Parameters	Data appeared in different reference			
The length of half-pipe	160-220	160	80	100-110

The width of flat	18	18	18	13-15
The height of platform	7~8	6	5	3-3.5
The inclined angle	18	18	15	15~18

Parameters	Data appeared in different reference			
The length of half-pipe	135	135	120	130
The width of flat	17.5	17.5	17.0	16.0
The height of platform	4.8	4.8	4.5	4.6
The inclined angle	16.5	16.5	16.0	16.5

A6. The results calculated in the solution

v_0	m/s	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5	12
a_x	m	124.31	128.72	135.12	139.78	144.25	148.47	152.45	156.15	159.1	162.94	165.94
h_x	m	2.50	2.89	3.27	3.69	4.13	4.60	5.10	5.63	6.17	6.75	7.35
w_x	m	15.22	15.73	16.23	16.73	17.23	17.73	18.24	18.74	19.24	19.74	20.24
δ	degree	40.69	40.79	44.67	46.03	47.17	48.29	49.39	50.49	51.58	52.64	53.70
$S_{\max}$	m	15.17	15.67	16.17	16.67	17.17	17.67	18.17	18.67	19.17	19.67	20.17
$v_{\max}$	m	19.25	19.75	20.25	20.75	21.25	21.75	22.25	22.75	23.25	23.75	24.25
v_s	m/s	17.28	17.62	17.91	18.23	18.55	18.86	19.17	19.46	19.76	20.05	20.34
t_t	s	2.30	2.35	2.57	2.68	2.78	2.87	2.97	3.06	3.16	3.25	3.35