第七章 多元函数微分学及其应用

一、选择题

1. 二元函数$z=\arcsin\frac{x}{2}+\frac{\sqrt{6x-y^2}}{\ln(1-x^2-y^2)}$的定义域为________.

答案：$D=\left\{(x,y)\middle|\ |x|\le 2,\ y^2\le 6x,\ 0<x^2+y^2<1\ \right\}$

解 因为$\begin{cases}\frac{|x|}{2}\le 1\\ y^2\le 6x\\ 0<x^2+y^2<1\end{cases}$ 即$\begin{cases}|x|\le 2\\ y^2\le 6x\\ 0<x^2+y^2<1\end{cases}$ 故定义域为：

$$D=\left\{(x,y)\middle|\ |x|\le 2,\ y^2\le 6x,\ 0<x^2+y^2<1\ \right\}$$

2. 求极限$\lim\limits_{(x,y)\to(0,0)}\frac{1-\cos\sqrt{x^2+y^2}}{\ln(x^2+y^2+1)}=$_________.

答案：0

解 $$\lim_{(x,y)\to(0,0)}\frac{1-\cos\sqrt{x^2+y^2}}{\ln(x^2+y^2+1)}=\lim_{(x,y)\to(0,0)}\frac{\frac{1}{2}(x^2+y^2)^2}{x^2+y^2}=0$$

3. 已知点$A(1,1,1)$及点$B(3,\ 2,\ -1)$，求函数$u=\ln(3xy-2z^3)$在点A处沿$\overrightarrow{AB}$的方向的方向导数为________.

答案： 7

解 $\overrightarrow{AB}=(2,1,-2)$ ，$\overrightarrow{AB}^o=\frac{1}{3}(2,1,-2)$

$$\frac{\partial u}{\partial l}=\left.\frac{\partial u}{\partial x}\right|_{(1,1,1)}\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{(1,1,1)}\cos\beta+\left.\frac{\partial u}{\partial z}\right|_{(1,1,1)}\cos\gamma$$

$$=\frac{2}{3}\cdot\left.\frac{3y}{3xy-2z^3}\right|_{(1,1,1)}+\frac{1}{3}\cdot\left.\frac{3x}{3xy-2z^3}\right|_{(1,1,1)}+(-\frac{2}{3})\cdot\left.\frac{-6z^2}{3xy-2z^3}\right|_{(1,1,1)}=7$$

4．函数$u=\ln(x^2+y^2+z^2)$在点$M(1,2,-2)$处的梯度$\mathbf{grad}\ u\big|_M=$_________.

答案：$\frac{2}{9}\vec{i}+\frac{4}{9}\vec{j}-\frac{4}{9}\vec{k}.$ 或$\frac{2}{9}\{1,2,-2\}$

解 $$\text{grad}\ u\big|_M=\left.\frac{\partial u}{\partial x}\right|_M\vec{i}+\left.\frac{\partial u}{\partial y}\right|_M\vec{j}+\left.\frac{\partial u}{\partial z}\right|_M\vec{k}$$

$$=\left.\frac{2x}{x^2+y^2+z^2}\right|_M\vec{i}+\left.\frac{2y}{x^2+y^2+z^2}\right|_M\vec{j}+\left.\frac{2z}{x^2+y^2+z^2}\right|_M\vec{k}$$

$$=\frac{2}{9}\vec{i}+\frac{4}{9}\vec{j}-\frac{4}{9}\vec{k}=\frac{2}{9}\{1,2,-2\}.$$

5. 若函数 $z=z(x,y)$ 由方程 $e^{z}=2-xyz-x-\cos x$ 确定，则 $dz\big|_{(0,1)}=$ ______ .

答案：$-dz$

解 令 $F(x,y,z)=e^{z}-2+xyz+x+\cos x$，求出 F_x,F_y,F_z,

则 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$, 由方程可知，当 $x=0,y=1$， $z=0$

所以 $dz\big|_{(0,1)}=-\frac{F_x}{F_z}dx+\frac{\partial z}{\partial y}dy=-dz$

二、选择题

1 二元函数 $z=f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0 & ,x^2+y^2=0\end{cases}$ 在 $(0,0)$ 点的邻域内偏导数 f_x' 及 f_y' 的存在情况为（ ）.

(A) 偏导数 f_x' 及 f_y' 都不存在； (B)偏导数 f_x' 及 f_y' 都存在；

(C) 偏导数 f_x' 存在但 f_y' 不存在； (D) 偏导数 f_y' 存在但 f_x' 不存在.

答案：选 B

解 偏导数定义可知：$f_x'(0,0)=f_y'(0,0)=0$

2、设 $z=\ln(\sqrt{x}+\sqrt{y})$, 则 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$ 的值为（ ）. .

(A) 不存在； (B) 1 ； (C) $\frac{1}{2}$ ； (D) 0 .

答案：选 C

解： $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x}+\sqrt{y}}\cdot\frac{1}{2\sqrt{x}}$， $\frac{\partial z}{\partial y}=\frac{1}{\sqrt{x}+\sqrt{y}}\cdot\frac{1}{2\sqrt{y}}$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}=x\frac{1}{\sqrt{x}+\sqrt{y}}\cdot\frac{1}{2\sqrt{x}}+y\frac{1}{\sqrt{x}+\sqrt{y}}\cdot\frac{1}{2\sqrt{y}}=\frac{1}{2}$$

3、函数 $z=\arctan\dfrac{y}{x}$ 的全微分（　　）

(A) $\dfrac{x\mathrm{d}x-y\mathrm{d}y}{x^2+y^2}$；　(B) $\dfrac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}$；　(C) $\dfrac{y\mathrm{d}x-x\mathrm{d}y}{x^2+y^2}$；　(D) $\dfrac{y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}$.

答案：选 B

解 $\dfrac{\partial z}{\partial x}=\dfrac{-\dfrac{y}{x^2}}{1+\dfrac{y^2}{x^2}}=\dfrac{-y}{x^2+y^2}$，　$\dfrac{\partial z}{\partial y}=\dfrac{-\dfrac{1}{x}}{1+\dfrac{y^2}{x^2}}=\dfrac{x}{x^2+y^2}$

所以　$dz=\dfrac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}$

4　设 $z=xy+xF(u)$，而 $u=\dfrac{y}{x}$，$F(u)$ 为可导函数，则 $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}$ 等于（　　）.

(A) $x-y+z$；　(B) $x+y+z$；　(C) $xy-z$；　(D) $xy+z$.

答案：D

解：$\dfrac{\partial z}{\partial x}=y+F(u)+xF'(u)\cdot(-\dfrac{y}{x^2})$，$\dfrac{\partial z}{\partial y}=x+xF'(u)\cdot\dfrac{1}{x}$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+xF(u)+xF'(u)\cdot(-\frac{y}{x})+yx+yxF'(u)\cdot\frac{y}{x}=xy+z$$

5．曲线 $\begin{cases} y=x^2 \\ z=x^2+y^2 \end{cases}$ 上点 $(1,1,2)$ 处的切线方程为（　　）.

A．$\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{8}$　　B．$\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{6}$

C．$x=\dfrac{y+1}{2}=\dfrac{z+4}{6}$　　D．$\dfrac{x-1}{1}=\dfrac{y-1}{-2}=\dfrac{z-2}{8}$

答案：选 B

解　曲线的切向量为 $\vec{l}=\{1,2x,2x+4x^3\}\big|_{x=1}=\{1,2,6\}$，因此，所求切线方程为

$\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z-2}{6}$.

三、解答题

1.　已知 $z=\ln(x+\sqrt{x^2+y^2})$，求 $\dfrac{\partial^2 z}{\partial x^2}$ 和 $\dfrac{\partial^2 z}{\partial x\partial y}$；

解： $\frac{\partial z}{\partial x}=\frac{1}{x+\sqrt{x^2+y^2}}\left(1+\frac{2x}{2\sqrt{x^2+y^2}}\right)=\frac{1+\frac{x}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}$

$$=\frac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}(x+\sqrt{x^2+y^2})}=\frac{1}{\sqrt{x^2+y^2}},$$

$\frac{\partial^2 z}{\partial x^2}=-\frac{2x}{2\sqrt{\left(x^2+y^2\right)^3}}=\frac{-x}{\left(x^2+y^2\right)^{\frac{3}{2}}}$， $\frac{\partial^2 z}{\partial x\partial y}=-\frac{2y}{2\sqrt{\left(x^2+y^2\right)^3}}=\frac{-y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$

2. 设 $z=u^2v-uv^2$，$u=x\cos y$，$v=x\sin y$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=(2uv-v^2)\cos y+(u^2-2uv)\sin y$$

$$=(2x^2\cos y\sin y-x^2\sin^2 y)\cos y+(x^2\cos^2 y-2x^2\sin y\cos y)\sin y$$

$$=3x^2\sin y\cos y(\cos y-\sin y),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=(2uv-v^2)(-x\sin y)+(u^2-2uv)x\cos y$$

$$=(2x^2\cos y\sin y-x^2\sin^2 y)(-x\sin y)+(x^2\cos^2 y-2x^2\sin y\cos y)x\cos y$$

$$=x^3(\sin^3 y+\cos^3 y-\sin y\sin 2y-\cos y\sin 2y).$$

3. 设 $u=f\left(\frac{x}{y},\frac{y}{z}\right)$，求 $\frac{\partial u}{\partial x}$， $\frac{\partial u}{\partial y}$， $\frac{\partial u}{\partial z}$.

解 $\frac{\partial u}{\partial x}=f_1'\cdot\frac{1}{y}=\frac{f_1'}{y}$， $\frac{\partial u}{\partial y}=f_1'\cdot\left(-\frac{x}{y^2}\right)+f_2'\cdot\frac{1}{z}=-\frac{x}{y^2}f_1'+\frac{1}{z}f_2'$，

$$\frac{\partial u}{\partial z}=f_2'\cdot\left(-\frac{y}{z^2}\right)=-\frac{y}{z^2}f_2'$$

4. 设 $z=f(y\sin x,\frac{y}{x},\mathrm{e}^{x+y})$，求 $\frac{\partial z}{\partial x}$，$\frac{\partial z}{\partial y}$.

解 令 $u=y\sin x$，$v=\frac{y}{x}$，$w=\mathrm{e}^{x+y}$，则 $z=f(u,v,w)$.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}\\ &= \frac{\partial f}{\partial u}\cdot y\cos x+\frac{\partial f}{\partial v}\cdot(-\frac{y}{x^2})+\frac{\partial f}{\partial w}\cdot \mathrm{e}^{x+y}\\ &= y\cos x\frac{\partial f}{\partial u}-\frac{y}{x^2}\frac{\partial f}{\partial v}+\mathrm{e}^{x+y}\frac{\partial f}{\partial w},\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}\\ &= \frac{\partial f}{\partial u}\cdot\sin x+\frac{\partial f}{\partial v}\cdot\frac{1}{x}+\frac{\partial f}{\partial w}\cdot \mathrm{e}^{x+y}\\ &= \sin x\cdot\frac{\partial f}{\partial u}+\frac{1}{x}\cdot\frac{\partial f}{\partial v}+\mathrm{e}^{x+y}\cdot\frac{\partial f}{\partial w}.\end{aligned}$$

5. 设 $z=z(x,y)$ 是由方程 $x^2+y^2+z^2=y\mathrm{e}^z$ 所确定的隐函数，求 dz .

解：设 $F(x,y,z)=x^2+y^2+z^2-y\mathrm{e}^z$，则 $F_x=2x$， $F_y=2y-e^z$， $F_z=2z-ye^z$，

故 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_x}=\frac{2x}{ye^z-2z}$， $\frac{\partial z}{\partial y}=-\frac{F_y}{F_x}=\frac{2y-e^z}{ye^z-2z}$

所以 $dz=\frac{2x}{ye^z-2z}dx+\frac{2y-e^z}{ye^z-2z}dy$

6. 设 $z=z(x,y)$ 由方程 $z+x=\mathrm{e}^{z-y}$ 所确定，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

解 对方程 $z+x=e^{z-y}$ 两边关于 x， y 求偏导，得 $\frac{\partial z}{\partial y}=e^{z-y}\left(\frac{\partial z}{\partial y}-1\right)$， $\frac{\partial z}{\partial x}+1=e^{z-y}\frac{\partial z}{\partial x}$

由此得 $$\frac{\partial z}{\partial y}=\frac{e^{z-y}}{e^{z-y}-1}=\frac{1}{1-e^{y-z}}，\ \frac{\partial z}{\partial x}=\frac{1}{e^{z-y}-1}.$$

7．设 $\varphi(u,v)$ 具有连续偏导数，函数 $z=f(x,y)$ 由方程 $\varphi(x-az,y-bz)=0$ 所确定，求 $a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}$.

解 令 $u=x-az$， $v=y-bz$，则

$$\varphi_x=\varphi_u\cdot\frac{\partial u}{\partial x}=\varphi_u，\ \varphi_y=\varphi_v\cdot\frac{\partial v}{\partial y}=\varphi_v，\ \varphi_z=\varphi_u\cdot\frac{\partial u}{\partial z}+\varphi_v\cdot\frac{\partial v}{\partial z}=-a\varphi_u-b\varphi_v.$$

$$\frac{\partial z}{\partial x}=-\frac{\varphi_x}{\varphi_z}=\frac{\varphi_u}{a\varphi_u+b\varphi_v}，\ \frac{\partial z}{\partial y}=-\frac{\varphi_y}{\varphi_z}=\frac{\varphi_v}{a\varphi_u+b\varphi_v}.$$

于是 $$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=a\cdot\frac{\varphi_u}{a\varphi_u+b\varphi_v}+b\cdot\frac{\varphi_v}{a\varphi_u+b\varphi_v}=1.$$

8. 证明曲面 $xyz=a^3$（$a>0$，为常数）的任一切平面与三个坐标面所围成的四面体的体积为常数.

证 设 $F(x,y,z)=xyz-a^3$，曲面上任一点 (x,y,z) 的法向量为 $n=(yz,xz,xy)$，该点的切平面方程为

$$yz(X-x)+xz(Y-y)+xy(Z-z)=0,$$

即 $$yzX+xzY+xyZ=3a^3.$$

这样，切平面与三个坐标面所围成的四面体体积为 $V=\frac{1}{6}\cdot\frac{3a^3}{yz}\cdot\frac{3a^3}{xz}\cdot\frac{3a^3}{xy}=\frac{9}{2}a^3$.

9. 求曲面 $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ 的一切平面，使其在三个坐标轴上的截距之积为最大.

解 曲面在点 (x,y,z) 处的法向量为 $\vec{n}=\left\{\frac{1}{2\sqrt{x}},\frac{1}{2\sqrt{y}},\frac{1}{2\sqrt{z}}\right\}$.

切平面方程为 $\frac{1}{\sqrt{x}}(X-x)+\frac{1}{\sqrt{y}}(Y-y)+\frac{1}{\sqrt{z}}(Z-z)=0$，即

$$\frac{X}{\sqrt{x}}+\frac{Y}{\sqrt{y}}+\frac{Z}{\sqrt{z}}=\sqrt{x}+\sqrt{y}+\sqrt{z}=1.$$

切平面在三坐标轴上的截距分别为 $\sqrt{x},\sqrt{y},\sqrt{z}$,根据题意，即要求在曲面 $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$ 上求一点 (x,y,z) 使 $f(x,y,z)=\sqrt{xyz}$ 取最大值. 为此，作拉格朗日函数

$$F(x,y,z)=xyz+\lambda(\sqrt{x}+\sqrt{y}+\sqrt{z}-1),$$

$$\begin{cases} F_x=yz+\dfrac{\lambda}{2\sqrt{x}}=0, & (1)\\ F_y=xz+\dfrac{\lambda}{2\sqrt{y}}=0, & (2)\\ F_z=xy+\dfrac{\lambda}{2\sqrt{z}}=0, & (3)\\ \sqrt{x}+\sqrt{y}+\sqrt{z}=1. & (4)\end{cases}$$

由(1)、(2)、(3)解得 $x=y=z$.代入(4)，得 $x=y=z=\frac{1}{9}$.依题意，$(\frac{1}{9},\frac{1}{9},\frac{1}{9})$ 即为所求点，曲面在该点的切平面在三坐标轴上的截距乘积为最大.该切平面方程为 $x+y+z=\frac{1}{3}$.

10. 在第一卦限内作椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 切平面，使该切面与三个坐标平面所围成的四面体体积最小，求切点的坐标.

解 设切点坐标为 $(x_0,\, y_0,\, z_0)$，切平面方程为 $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$，四面体体积为 $V=\frac{a^2b^2c^2}{6x_0y_0z_0}$，

设 $F(x,y,z,\lambda)=xyz+\lambda(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1)$，令

$$\begin{cases} F_x=yz+2\lambda\frac{x}{a^2}=0, \\ F_y=xz+2\lambda\frac{y}{b^2}=0 \\ F_z=xy+2\lambda\frac{z}{c^2}=0 \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \end{cases}$$

，解得切点坐标为 $(\frac{\sqrt{3}}{3}a, \frac{\sqrt{3}}{3}b, \frac{\sqrt{3}}{3}c)$.