

二、1.

(1) 证: $f(x) = \sum_{n=0}^{+\infty} \frac{1}{3^n + x} \leq \sum_{n=0}^{+\infty} \frac{1}{3^n}$, 故 $\sum_{n=0}^{+\infty} \frac{1}{3^n + x}$ 收敛

$f(x)$ 连续, $\forall x_1, x_2 \in (0, +\infty)$

$$|f(x_1) - f(x_2)| = \sum_{n=0}^{+\infty} \frac{1}{(3^n + x_1)(3^n + x_2)} |x_1 - x_2| \leq \frac{1}{3^n} |x_1 - x_2|$$

$f(x)$ 一致连续, $|(\sum_{n=0}^{+\infty} \frac{1}{3^n + x})'| \leq \frac{1}{3^n}$, $\sum_{n=0}^{+\infty} (\frac{1}{3^n + x})'$ 一致收敛

所以 $f(x)$ 在 $(0, +\infty)$ 可导

(2) 证: $\int_0^{+\infty} f(x) dx = \sum_{n=0}^{+\infty} \int_0^{+\infty} \frac{1}{3^n + x} dx$

$$= \lim_{x \rightarrow +\infty} \sum_{n=0}^{+\infty} (\ln(x + 3^n) - \ln 3^n)$$

$$= \lim_{x \rightarrow +\infty} \sum_{n=0}^{+\infty} \ln(1 + \frac{x}{3^n})$$

x 取子列 3^n 时, $\lim_{x \rightarrow +\infty} \sum_{n=0}^{+\infty} \ln 2$ 发散

故 $\int_0^{+\infty} f(x) dx$ 发散.

2.

证: $f'(t) = 2e^{-t} \int_0^t e^{-x^2} dx$, 令 $x = ty$

$$f'(t) = 2te^{-t^2} \int_0^1 e^{-t^2 y^2} dy$$

$$g'(t) = \int_0^1 \left(\frac{e^{-(1+x^2)t}}{1+x^2} \right)' dx = -2te^{-t^2} \int_0^1 e^{-t^2 x^2} dx$$

$$f'(t) + g'(t) = 2te^{-t^2} \int_0^1 e^{-t^2 y^2} dy - 2te^{-t^2} \int_0^1 e^{-t^2 x^2} dx = 0$$

因为 $f'(t) + g'(t) = 0$

故 $f(t) + g(t)$ 为常数

三、解:

1) 不可以

$$\text{令 } f(x) = \begin{cases} 1 & x = [x] \quad (\text{整数}) \\ 0 & x \neq [x] \quad (\text{非整}) \end{cases}$$

$$\int_0^{+\infty} f(x) dx = 0 \quad \text{收敛}$$

$$\text{但 } \lim_{x \rightarrow \infty} f(x) = 0 \quad \text{不成立}$$

2) ① $f(x)$ 单调

$$\int_a^{+\infty} f(x) dx \text{ 收敛, } f(x) \text{ 在 } [a, +\infty) \text{ 上有界}$$

有界单调则必有极限

$$\text{设 } \lim_{x \rightarrow \infty} f(x) = T \neq 0$$

$$\text{则 } \exists N > 0, \text{ 使 } a' > N, |f(x) - T| < \varepsilon$$

$$\int_{a'}^{+\infty} f(x) dx \text{ 发散, } \int_a^{+\infty} f(x) dx \text{ 发散}$$

$$\text{矛盾, 所以 } \lim_{x \rightarrow \infty} f(x) = 0$$

② $\int_a^{+\infty} f'(x) dx$ 收敛

$$\int_a^{+\infty} f'(x) dx = \lim_{x \rightarrow \infty} f(x) - f(a)$$

$$\lim_{x \rightarrow \infty} f(x) \text{ 存在, 若 } \lim_{x \rightarrow \infty} f(x) = T \neq 0$$

$$\text{则 } \int_a^{+\infty} f(x) dx \text{ 发散, 矛盾}$$

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课程名称: 数学分析2 学号: 1101190111 姓名: 唐川淇 班级: 信计1901

1. 解: $\frac{\partial z}{\partial x} = -\sin x f'(\cos x - \cos y)$, $\frac{\partial z}{\partial y} = -\sin y + \sin y f'(\cos x - \cos y)$
$$\csc x \frac{\partial z}{\partial x} + \csc y \frac{\partial z}{\partial y}$$
$$= -f' - 1 + f'$$
$$= -1$$

2. 解: $\int_0^1 x(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \dots) dx$
$$= \int_0^1 x e^{x^2} dx$$
$$= \frac{1}{2} e^{x^2} \Big|_0^1$$
$$= \frac{e}{2} - \frac{1}{2}$$

3. 解: $\lim_{(x,y) \rightarrow (0,0)} (a\sqrt{x} + 2x^2 + y^2 + b) \frac{\tan xy^2}{x^2 + y^4}$
$$\leq (a\sqrt{x} + 2x^2 + y^2 + b) \left| \frac{\tan xy^2}{2|xy^2|} \right|$$
$$\leq \frac{1}{2} (a\sqrt{x} + 2x^2 + y^2 + b)$$

令 $x = \rho \cos \theta$
 $y = \rho \sin \theta$

原式 = $\lim_{\rho \rightarrow 0} (a\sqrt{\rho \cos \theta} + \rho^2 + \rho^2 \cos \theta + b) = 0$

故 $b = 0$, a 为任意常数

4. 解: (1) $h = 75 - x^2 - y^2 + xy$

$$\text{grad } h(x, y) = \frac{\partial}{\partial x} h(x, y) \vec{i} + \frac{\partial}{\partial y} h(x, y) \vec{j}$$

$$\frac{\partial h}{\partial x} = -2x + y, \quad \frac{\partial h}{\partial y} = -2y + x$$

$$|\text{grad } h(x, y)| = \sqrt{(-2x + y)^2 + (-2y + x)^2}$$

$M(x_0, y_0)$ 处方向导数最大

$$g(x_0, y_0) = |\text{grad } h(x, y)|_{(x_0, y_0)}$$

$$\begin{aligned} g(x_0, y_0) &= \sqrt{(y_0 - 2x_0)^2 + (x_0 - 2y_0)^2} \\ &= \sqrt{5x_0^2 + 5y_0^2 - 8x_0y_0} \end{aligned}$$

(2) $L = 5x_0^2 + 5y_0^2 - \lambda(x^2 + y^2 - xy - 75)$

$$\begin{cases} L_x = 10x - 8y - 2\lambda x - \lambda y = 0 \\ L_y = 10y - 8x - 2\lambda y - \lambda x = 0 \\ L_\lambda = x^2 + y^2 - xy - 75 = 0 \end{cases}$$

解得 $\begin{cases} x_1 = 5 \\ y_1 = -5 \end{cases} \begin{cases} x_2 = -5 \\ y_2 = 5 \end{cases} \begin{cases} x_3 = 5\sqrt{3} \\ y_3 = 5\sqrt{3} \end{cases} \begin{cases} x_4 = -5\sqrt{3} \\ y_4 = -5\sqrt{3} \end{cases}$

分别代入 $g^2(x, y)$

$$g_1^2 = 450, \quad g_2^2 = 450, \quad g_3^2 = 150, \quad g_4^2 = 150$$

故 $M_1(5, -5), M_2(-5, 5)$ 可为起点,

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