

graph

First, perform bucket sorting on c_i to transform it into a strictly monotonically increasing sequence of positive integers. Now all the following c_i satisfy this assumption.

For each vertex i , we split it into two points i^+ and i^- , with one representing the incoming edge and the other representing the outgoing edge. For example, if there is an edge from 1 to 2, it becomes an edge between 1^- and 2^+ . It is not difficult to see that the resulting graph is an undirected bipartite graph.

Let's first ignore the conditions of self-loops, double edges, and bidirectional edges. Then the problem becomes: for an undirected bipartite graph $G = X + Y$, find $|E(G)|_{\min}$ such that there exist m points with degrees of $c_1, c_2, \dots, \text{and } c_m$ respectively.

In fact, we consider the points with degrees $c_{m-i+1}, \dots, c_{m-1}, c_m$. Suppose there are a points inside X and b points inside Y ; then $a + b = i$. The set $E(G)$ contains all the edges formed by these points, and the number of these edges is $\geq \sum_{k=m-i+1}^m c_k - ab$ (total minus the repeated counts) $\geq \sum_{k=m-i+1}^m c_k - \left\lfloor \frac{i^2}{4} \right\rfloor$.

Hence, we obtain a lower bound for $|E(G)|$:

$$\max_{1 \leq i \leq m} \left\{ \sum_{k=m-i+1}^m c_k - \left\lfloor \frac{i^2}{4} \right\rfloor \right\}$$

Next, we construct an example that can obtain this many edges. Let $X = \{u_1, \dots, u_n\}$ and $Y = \{v_1, \dots, v_n\}$. Assume that the expression reaches its maximum value at $i = t$. If the maximum is achieved at multiple points, choose the one with the smallest t . Let $a = \left\lceil \frac{t}{2} \right\rceil$ and $b = \left\lfloor \frac{t}{2} \right\rfloor$.

We select a points from set X , with the degrees of these points being $c_m, c_{m-2}, \dots, c_{m-2(a-1)}$, and we select b points from set Y , with the degrees of these points being $c_{m-1}, c_{m-3}, \dots, c_{m-(2b-1)}$. Let these points be $u_1, \dots, u_a, v_1, \dots, v_b$.

Our construction method is as follows: For all $1 \leq j \leq a$, u_j is connected to $v_1, v_2, \dots, v_{c_{m-2j+2}}$; for all $1 \leq k \leq b$, v_k is connected to $u_1, u_2, \dots, u_{c_{m-2k+1}}$.

Since the expression reaches its maximum value when $i = t$, we have $c_{m-t} \leq \left\lceil \frac{t}{2} \right\rceil$ (comparing the two expressions when $i = t$ and $i = t + 1$), and $c_{m-t+1} \geq \left\lfloor \frac{t+1}{2} \right\rfloor$ (comparing when $i = t$ with $i = t - 1$ and considering the minimality of t). Hence, in this construction, points with degrees of $1, 2, \dots, \left\lceil \frac{t}{2} \right\rceil$ are necessarily present. Therefore, this construction meets the requirements. Additionally, we observe that when $(u_1, u_2, \dots, u_n) = (1, 2, \dots, n)$ and $(v_1, v_2, \dots, v_n) = (n, n-1, \dots, 1)$, this construction can directly handle the “no self-loops, multiple edges, and bidirectional edges” condition in the problem. In fact, it can be noticed that in this construction, there do not exist $1 \leq p, q \leq n$ such that p^- is connected to q^+ and p^+ is also connected to q^- .

Over. When $opt = 0$, the time complexity is $\mathcal{O}(n^2)$; when $opt = 1$, the time complexity is $\mathcal{O}(n + m)$.